International Journal of Theoretical Physics, Vol. 35, No. 6, 1996

Semi-Orthoposets

Stanley Gudder[1]

Received September 26, 1995

A semi-orthoposet (SOP) is a bounded poset P together with a unary operation $'$: $P \rightarrow P$ such that $a \leq b$ implies $b' \leq a'$ and $a \leq a''$ for all $a \in P$. This structure generalizes all previously studied quantum logic frameworks and yet is rich enough so that nontrivial results can be proved. For various types of SOPs it is shown that a partially defined morphism has an extension to the full SOP and that there exists an order-determining set of morphisms with a specified range. These results are applied to obtain representations of SOPs in terms of SOPs of sets and SOPs of functions. Connections between SOPs and effect algebras as well as tensor products of SOPs are obtained.

1. INTRODUCTION

Various ordered structures have been considered in the quantum logic approach to the foundations of quantum physics. Sixty years ago, Birkhoff and von Neumann (1936) originated the quantum logic approach by proposing the framework of a modular, complemented lattice. This framework was later generalized to orthomodular lattices and posets (Beltrametti and Cassinelli, 1981; Gudder, 1979, 1988; Jauch, 1968; Mackey, 1963; Piron, 1976; Ptak and Pulmannová, 1991; Varadarajan, 1968/1970). Unfortunately, the categories of orthomodular lattices and posets do not admit tensor products, which are important for the study of composite physical systems (Foulis and Randall, 1981; Kläy *et al.*, 1987; Randall and Foulis, 1981). For this and other reasons, a more general category has been recently introduced, that of orthoalgebras (Foulis and Bennett, 1993; Foulis *et al.*, 1992; Gudder, 1988). But orthoalgebras were still not general enough to include the study of unsharp measurements and propositions. This latter study is usually called the operational or convexity approach (Ludwig, 1986; Prugovečki, 1986; Schroeck and Foulis,

[1] Department of Mathematics and Computer Science, University of Denver, Denver, Colorado 80208.

0020-7748/96/0600-1141$09.50/0

1990). The latest framework that has been proposed is the category of effect algebras or difference posets (Dvurečenskij, 1995, n.d.; Dvurečenskij and Riečan, 1994; Foulis and Bennett, 1994, n.d.; Giuntini and Greuling, 1989; Kôpka and Chovanec, 1994). This framework includes all the previous ones and still admits tensor products. Other frameworks have been developed to study unsharp measurements. These include paraconsistent quantum logic, Brouwer–Zadeh posets, and three-valued Lukasiewicz posets (Cattaneo and Nistico, 1989; Dalla Chiara and Giuntini, 1989; Giuntini, 1990; Giuntini and Greuling, 1989).

In this article, we attempt to set an upper bound for this proliferating sequence of generalizing frameworks. The structure that we shall study is called a *semi-orthoposet* (SOP). This structure generalizes all the previously mentioned ones, and yet is rich enough so that nontrivial results can be proved. Besides this, it can be employed to generate orthoalgebras and effect algebras in a canonical fashion.

The article begins with basic definitions and results. It then illustrates various types of SOPs with examples. Our main concern is that of SOP morphisms, and these are studied next. The two main problems investigated are whether a partially defined morphism has an extension to the full SOP and whether there exists an order-determining set of morphisms with a specified range. We are able to solve both of these problems for various types of SOPs, but slightly different proofs must be used for each type. We then apply these results to obtain representation theorems for certain types of SOPs in terms of SOPs of sets and SOPs of functions. Furthermore, these results are employed to obtain tensor products for various types of SOPs. Finally, we give connections between SOPs and effect algebras.

One of the advantages of studying SOPs is that they unify the various stronger structures and give a deeper understanding of their properties. This is because we can define certain local properties that hold globally in the stronger structures.

2. BASIC DEFINITIONS AND RESULTS

A *semi-orthoposet* (SOP) is a bounded poset $(P, 0, 1, \leq)$ with a map $'$: $P \to P$ such that (1) $a \leq b$ implies $b' \leq a'$ and (2) $a \leq a''$ for all $a \in P$. A *unit* SOP is a SOP in which $1' = 0$. An element $a \in P$ is *complementing* if $a \wedge a' = 0$, *sharp* if $a \vee a' = 1$, and *closed* if $a = a''$. If $a \leq b'$, we write $a \perp b$. Notice that if $a \leq b'$, then $b \leq b'' \leq a'$, so $\perp$ is a symmetric relation. An element $a \in P$ is *isotropic* if $a \perp a$, *strongly isotropic* if $a = a'$, and *orthoisotropic* if $a \perp a$ and whenever $b \perp b$, then $a \perp b$. Also, a is *nonisotropic* if $a \not\perp a$.

The following lemmas contain useful basic results. The proof of each result is elementary but is included for completeness.

Lemma 2.1. A SOP P has the following properties. (i) $a''' = a'$ for all $a \in P$. (ii) $0' = 1$. (iii) If $\vee a_\alpha$ exists, then $\wedge a'_\alpha$ exists and $(\vee a_\alpha)' = \wedge a'_\alpha$. (iv) If $\wedge a_\alpha$ and $\vee a'_\alpha$ exist, then $\vee a'_\alpha \leq (\wedge a_\alpha)'$. (v) If P is a unit SOP and if a is sharp, then a is complementing. (vi) 1 is sharp and closed and 0 is sharp and complementing. (vii) The following are equivalent: P is a unit SOP, 0 is closed, 1 is complementing. (viii) If a is sharp, then a' is sharp. If a' is complementing, then a is complementing. (ix) An element a is closed if and only if $a = b'$ for some $b \in P$. (x) For $a \in P$, a'' is the smallest closed upper bound of a. (xi) If a is closed, then a is strongly isotropic if and only if a and a' are isotropic. (xii) If a is isotropic and $b \leq a$, then b is isotropic. (xiii) If $a \neq 0$ is complementing, then a is nonisotropic. (xiv) If a_α are closed and $\wedge a_\alpha$ exists, then $\wedge a_\alpha$ is closed.

Proof. (i) Since $a \leq a''$, $a''' \leq a'$. Also, $a' \leq (a')'' = a'''$. (ii) Since $0 \leq 1'$, $1 \leq 1'' \leq 0'$. Hence, $1 = 1'' = 0'$. (iii) Suppose $\vee a_\alpha$ exists. Then $a_\alpha \leq \vee a_\alpha$ for all α, so $(\vee a_\alpha)' \leq a'_\alpha$ for all α. Suppose $b \leq a'_\alpha$ for all α, Then $a_\alpha \leq a''_\alpha \leq b'$ for all α, so $\vee a_\alpha \leq b'$. Hence, $b \leq b'' \leq (\vee a_\alpha)'$, so $(\vee a_\alpha)' = \wedge a'_\alpha$. (iv) Since $\wedge a_\alpha \leq a_\alpha$ for all α, $a'_\alpha \leq (\wedge a_\alpha)'$ for all α. Hence, $\vee a'_\alpha \leq (\wedge a_\alpha)'$. (v) Suppose P is a unit SOP and a is sharp. Then $a \vee a' = 1$ and by (iii), $a' \wedge a'' = 1' = 0$. If $b \leq a, a'$, then $b \leq a'', a'$, so $b \leq a'' \wedge a' = 0$. Hence, $b = 0$, so $a \wedge a' = 0$ and a is complementing. (vi) We have shown in (ii) that 1 is closed. Since $1' \leq 1$, $1 \vee 1' = 1$, so 1 is sharp. Since $0 \leq 0'$, $0 \wedge 0' = 0$, so 0 is complementing. Since $0 \vee 0' = 0 \wedge 1 = 1$, 0 is sharp. (vii) If P is a unit SOP, then $0 = 1' = 1''' = 0''$, so 0 is closed. If 0 is closed, then $1' = 0'' = 0$, so $1 \wedge 1' = 1 \wedge 0 = 0$ and 1 is complementing. If 1 is complementing, then $1' = 1 \wedge 1' = 0$, so P is a unit SOP. (viii) If a is sharp, then $a \vee a' = 1$. If $b \geq a', a''$, then $b \geq a', a$, so $b \geq a \vee a' = 1$, so $b = 1$. Hence, $a' \vee a'' = 1$ and a' is sharp. If a' is complementing, then $a' \wedge a'' = 0$. If $b \leq a, a'$, then $b \leq a'', a'$, so $b \leq a' \wedge a'' = 0$. Hence, $a \wedge a' = 0$ and a is complementing. (ix) If a is closed, then $a = (a')'$. If $a = b'$ for some $b \in P$, then $a'' = b''' = b' = a$, so a is closed. (x) Clearly, a'' is a closed upper bound for a. If $a \leq b$ where b is closed, then $a'' \leq b'' = b$. (xi) Notice that a is strongly isotropic if and only if $a \leq a'$ and $a' \leq a = a''$. But this is equivalent to a and a' being isotropic. (xii) If a is isotropic and $b \leq a$, then $b \leq a \leq a' \leq b'$, so b is isotropic. (xiii) Suppose $a \neq 0$ is isotropic. Then $a \wedge a' = a \neq 0$, so a is not complementing. (xiv) Suppose a_α are closed and $\wedge a_\alpha$ exists. Then $\wedge a_\alpha \leq a_\alpha$ for all α, so $(\wedge a_\alpha)'' \leq a''_\alpha = a_\alpha$ for all α. Hence, $(\wedge a_\alpha)'' \leq \wedge a_\alpha$. It follows that $\wedge a_\alpha = (\wedge a_\alpha)''$, so $\wedge a_\alpha$ is closed. ∎

There are examples that show that the full De Morgan's law of (iv) does not hold. The converses of (v) and (xiii) do not hold. Part (xiv) does not hold if $\wedge$ is replaced by $\vee$.

The previous definitions give local properties of individual elements. We now give some global definitions. A SOP that is a lattice is called a SOL. A SOP P is *complementing, sharp,* or *closed* if every $a \in P$ is complementing, sharp, or closed, respectively. A SOP P is *nonisotropic* if every $0 \neq a \in P$ is nonisotropic and P is *regular* if every isotropic $a \in P$ is orthoisotropic. It is easy to show that a closed or complementing SOP is a unit SOP. A closed, sharp SOP is called an *orthoposet.*

Lemma 2.2. (i) The following statements are equivalent: P is closed, $'$ is injective, $'$ is surjective. (ii) If P is closed and $\wedge a_\alpha$ exists, then $\vee a'_\alpha$ exists and $(\wedge a_\alpha)' = \vee a'_\alpha$. (iii) If P is closed, then $a \in P$ is sharp if and only if a is complementing. (iv) A SOP P is nonisotropic if and only if P is complementing.

Proof. (i) Applying Lemma 2.1(ix), P is closed if and only if $'$ is surjective. If P is closed and $a' = b'$, then $a = a'' = b'' = b$, so $'$ is injective. If $'$ is injective, then $a' = (a'')'$ implies that $a = a''$, so P is closed. (ii) Suppose P is closed and $\wedge a_\alpha$ exists. Then $\wedge a_\alpha \le a_\alpha$ for all α, so $a'_\alpha \le (\wedge a_\alpha)'$ for all α. Suppose $a'_\alpha \le b$ for all α. Then $b' \le a''_\alpha = a_\alpha$ for all α, so $b' \le \wedge a_\alpha$. Hence, $(\wedge a_\alpha)' \le b'' = b$, so $(\wedge a_\alpha)' = \vee a'_\alpha$. (iii) If P is closed, then P is a unit SOP. Applying Lemma 2.1(v), if a is sharp, then a is complementing. Conversely, if a is complementing, then $a \wedge a' = 0$. Applying (ii) gives

$$1 = 0' = (a \wedge a')' = a' \vee a'' = a' \vee a$$

so a is sharp. (iv) Suppose P is complementing and $a \perp a$. Then $a \le a'$, so $a = a \wedge a' = 0$. Hence, P is nonisotropic. Conversely, suppose P is nonisotropic. Let $a \in P$ and suppose $b \le a, a'$. Then $b \le a \le a'' \le b'$, so b is isotropic. Hence, $b = 0$, so $a \wedge a' = 0$ and P is complementing. ■

One of the reasons for the terminology "a is sharp" is that $a \vee a' = 1$ is equivalent to the law of the excluded middle. We now give a counterexample that illustrates various results that do not hold. Let $P = \{0, 1, a, b, c, a', b', c'\}$, where $a'' = a$, $b'' = b$, $c'' = c$, $1' = 0$, $c < a, b, c'$, and $a', b' < c'$. It is easy to show that P is a closed SOP. Although a and b are complementing, sharp, and nonisotropic, $c = a \wedge b$ is not complementing, sharp, or nonisotropic. Similarly, a' and b' are complementing and sharp, but $c' = a' \vee b'$ is not complementing or sharp. Notice that it follows from Lemma 2.1(xii) that if a is a nonisotropic element of an arbitrary SOP and if $a \vee b$ exists, then $a \vee b$ is nonisotropic.

If P and Q are SOPs, a *morphism* ϕ: $P \to Q$ satisfies the following: $\phi(0) = 0$, $a \le b$ implies $\phi(a) \le \phi(b)$, and $\phi(a') = \phi(a)'$ for all $a \in P$. We sometimes call ϕ a *Q-morphism* on P. If, in addition, ϕ is bijective and $\phi(a) \leq \phi(b)$ implies $a \leq b$, then ϕ is an *isomorphism*. A set of Q-morphisms M on P is *order determining* if $m(a) \leq m(b)$ for all $m \in M$ implies $a \leq b$. A *sub*-SOP P_1 of a SOP P is a subset of P such that $0, 1 \in P_1$, and if $a \in P_1$, then $a' \in P_1$. Notice that a sub-SOP is itself a SOP. A *partial Q-morphism* on P is a Q-morphism defined on a sub-SOP of P.

An *effect algebra* is a system $(P, 0, 1, \oplus)$ where $\oplus$ is a partial binary operation on P satisfying:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.
(2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(3) For every $a \in P$ there exists a unique $a' \in P$ such that $a \oplus a' = 1$.
(4) If $a \oplus 1$ is defined, then $a = 0$.

For an effect algebra P, we write $a \perp b$ if $a \oplus b$ is defined. Moreover, we write $a \leq b$ if there exists a $c \in P$ such that $c \perp a$ and $b = a \oplus c$. If P and Q are effect algebras, an *effect morphism* is a map $\phi: P \to Q$ such that $\phi(1) = 1$ and $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An *orthoalgebra* is an effect algebra in which $a \perp a$ implies $a = 0$. The next lemma shows that the notation $a \perp b$ is consistent with our previous usage and that a closed SOP is a generalization of an effect algebra. This result can be found in Foulis and Bennett (1993, 1994).

Lemma 2.3. (i) If $(P, 0, 1, \oplus)$ is an effect algebra, then $a \perp b$ if and only if $a \leq b'$ and $(P, 0, 1, \leq, ')$ is a closed SOP. (ii) If $(P, 0, 1, \oplus)$ is an orthoalgebra, then $(P, 0, 1, \leq, ')$ is an orthoposet.

Proof. (i) If $a \perp b$, then $b \oplus a$ is defined. Now $(b \oplus a) \oplus (b \oplus a)' = 1$, so $b \oplus [a \oplus (b \oplus a)'] = 1$. Hence, $a \oplus (b \oplus a)' = b'$, so $a \leq b'$. Conversely, if $a \leq b'$, then there exists a $c \in P$ such that $a \perp c$ and $a \oplus c = b'$. Since $b \perp b'$, we conclude that $b \oplus (a \oplus c)$ is defined. Hence, $b \oplus a$ is defined so $a \perp b$. To show that $(P, 0, 1, \leq, ')$ is a closed SOP, since $a \oplus a' = 1$, we conclude that $a \leq 1$ for all $a \in P$. Since $1' \oplus 1$ is defined, $1' = 0$. Since $a' \oplus a = 1$, we have $a = a''$. It follows that

$$(0 \oplus a) \oplus a' = 0 \oplus (a \oplus a') = 0 \oplus 1 = 1$$

so $0 \oplus a = a'' = a$. Hence, $0 \leq a$ and $a \leq a$ for all $a \in P$. Suppose $a \leq b$ and $b \leq a$. Then there exist $c, d \in P$ such that $b = a \oplus c$ and $a = b \oplus d$. Hence,

$$b = (b \oplus d) \oplus c = b \oplus (d \oplus c)$$

We conclude that

$$1 = b' \oplus b = b' \oplus [b \oplus (d \oplus c)] = (b' \oplus b) \oplus (d \oplus c) = 1 \oplus (d \oplus c)$$

so $d \oplus c = 0$. Hence,

$$d' = d' \oplus 0 = d' \oplus (d \oplus c) = (d' \oplus d) \oplus c = 1 \oplus c$$

so $c = 0$. It follows that $b = a$. Suppose that $a \leq b$ and $b \leq c$. Then there exist $d, e \in P$ such that $b = a \oplus d$ and $c = b \oplus e$. Hence,

$$c = (a \oplus d) \oplus e = a \oplus (d \oplus e)$$

so $a \leq c$. Finally, if $a \leq b$, then there exists $c \in P$ such that $b = a \oplus c$. Hence,

$$1 = b \oplus b' = (a \oplus c) \oplus b' = a \oplus (c \oplus b')$$

We conclude that $a' = c \oplus b'$, so $b' \leq a'$. (ii) If a is isotropic, then $a \perp a$, so $a = 0$. It follows that P is nonisotropic, so by Lemma 2.2(iv), P is complementing. By Lemma 2.2(iii), P is sharp, so P is an orthoposet. ∎

Of course, an effect morphism $\phi: P \to Q$ is a SOP morphism when P and Q are considered as SOPs.

3. EXAMPLES

This section illustrates the generality and unifying power of SOPs by exhibiting a large number of examples.

Example 1. Let X be a nonempty set and let r be a symmetric relation on X. For $A \in 2^X$, define

$$A' = \{y \in X: y\, r\, x \text{ for all } x \in A\} \tag{3.1}$$

Then $P = (2^X, \emptyset, X, \subseteq, ')$ is a SOL. Simple examples show that P need not be a unit SOL and P need not be complementing, sharp, or closed. If r is also irreflexive, then P is a complementing SOL that need not be sharp or closed. Conversely, if P is complementing, then r is irreflexive. Indeed, in this case $\{x\} \cap \{x\}' = \emptyset$, so x is not related to x for all $x \in X$. If $P_0 = \{A \in P: A = A''\}$, then it is easy to show that P_0 is a complete ortholattice. A *set*-SOP is a SOP of the form $(P, \emptyset, X, \subseteq, ')$ where $P \subseteq 2^X$. [In this definition $'$ is general and need not have the form (3.1).]

Lemma 3.1. $(2^X, \emptyset, X, \subseteq, ')$ is a set-SOL if and only if there is a symmetric relation r on X such that A' has the form (3.1) for all $A \in 2^X$.

Proof. We have already noted sufficiency. For necessity, suppose $(2^X, \emptyset, X, \subseteq, ')$ is a set-SOL. Define $y\, r\, x$ if $y \in \{x\}'$. If $y\, r\, x$, then $\{y\} \subseteq \{x\}'$, so $\{x\} \subseteq \{x\}'' \subseteq \{y\}'$. Hence, $x\, r\, y$, so r is a symmetric relation. For $A \in 2^X$, since $A = \cap_{x \in A} \{x\}$, by De Morgan's law we have

$$A' = \bigcap_{x \in A} \{x\}' = \{y \in X: y\, r\, x \text{ for all } x \in A\} \quad ∎$$

A set-SOP $(P, \varnothing, X, \subseteq, ')$ is *generated* by a symmetric relation r if A' has the form (3.1) for all $A \in P$. By Lemma 3.1, $(2^X, \varnothing, X, \subseteq, ')$ is always generated by a symmetric relation.

Lemma 3.2. A set-SOP $(P, \varnothing, X, \subseteq, ')$ is generated by a symmetric relation if and only if for every $A \in P$ we have

$$\bigcap_{x \in A} \cup \{B' : x \in B \in P\} \subseteq A'$$

Proof. Suppose $(P, \varnothing, X, \subseteq, ')$ is generated by a symmetric relation r on X. If $B \in P$, $y \in B'$, and $x \in B$, then $y\ r\ x$. Hence if

$$y \in \bigcap_{x \in A} \cup \{B' : x \in B \in P\}$$

then $y\ r\ x$ for all $x \in A$, so $y \in A'$. Conversely, let $(P, \varnothing, X, \subseteq, ')$ be a set-SOP satisfying the given condition. Define $x\ r\ y$ if there exists a $B \in P$ such that $x \in B$ and $y \in B'$. If $x\ r\ y$, then $y \in B'$ and $x \in B \subseteq (B')'$, so $y\ r\ x$. Hence, r is a symmetric relation on X. If $y\ r\ x$ for every $x \in A$, then by the condition, $y \in A'$. If $y \in A'$, then for every $x \in A$ we have $y\ r\ x$, so A' has the form (3.1). ■

The next lemma shows that any SOP can be represented as a set-SOP.

Lemma 3.3. Any SOP Q is isomorphic to a set-SOP.

Proof. For $a \in Q$, define $(0, a] \subseteq Q$ by

$$(0, a] = \{b \in Q: 0 < b \leq a\}$$

Let $X = (0, 1]$, $P = \{(0, a]: a \in Q\}$ and define $(0, a]' = (0, a']$. Then $P \subseteq 2^X$ and we now show that $(P, \varnothing, X, \subseteq, ')$ is a set-SOP. Since $(0, 0] = \varnothing$, we have $\varnothing \in P$ and it is clear that $(0, a] \subseteq (0, b]$ if and only if $a \leq b$. Hence, if $(0, a] \subseteq (0, b]$, then $b' \leq a'$, so

$$(0, b]' = (0, b'] \subseteq [0, a'] = (0, a]'$$

Moreover, since $a \leq a''$, we have

$$(0, a] \subseteq (0, a''] = (0, a]''$$

Define $\phi: Q \to P$ by $\phi(a) = (0, a]$. It is clear that ϕ is an isomorphism. ■

Example 2. Let $L_0(X)$ be the lattice of open subsets of a topological space X. For $A \in L_0(X)$, define $A' = \mathrm{int}(A^c)$. Then $(L_0(X), \varnothing, X, \subseteq, ')$ is a complementing set-SOL that is not sharp and not closed in general. It is easy to show that $A \in L_0(X)$ is sharp if and only if A is clopen. Moreover, if A is sharp, then A is closed ($A = A''$). The next lemma shows that, in general, this set-SOL is not generated by a symmetric relation on X.

Lemma 3.4. If X is a Hausdorff space that contains an open set A that is not topologically closed, then $(L_0(X), \varnothing, X, \subseteq, ')$ is not generated by a symmetric relation on X.

Proof. If $x, y \in X$ with $x \neq y$, then there exist disjoint open sets U and V such that $x \in U$, $y \in V$. Hence, $y \in U'$ and

$$\{x\}^c = \cup\{B' \colon x \in B \in L_0(X)\}$$

It follows that

$$A^c = \left(\bigcup_{x \in A} \{x\}\right)^c = \bigcap_{x \in A} \{x\}^c = \bigcap_{x \in A} \cup \{B' \colon x \in B \in L_0(X)\}$$

But $A^c \not\subseteq A'$ since A is not topologically closed. Applying Lemma 3.2 gives the result. ∎

Example 3. Let V be a real or complex inner product space with inner product $\langle x, y\rangle$ and let $L(V)$ be the lattice of all subspaces of V. Define the symmetric relation $x \perp y$ on V by $\langle x, y\rangle = 0$ and for $A \in L(V)$ define

$$A' = \{y \in V \colon y \perp x \text{ for all } x \in A\}$$

Then $(L(V), \{0\}, V, \subseteq, ')$ is a complementing SOL that is not sharp and not closed, in general. It is easy to show that $A \in L(V)$ is sharp if and only if $A + A' = V$ and that the set of sharp subspaces forms an orthoposet.

Example 4. Let $S = [0, 1] \subseteq \mathbb{R}$ with the usual order $\leq$. For $a \in S$, define $a' = 1 - a$. Then S is a closed SOL that is not complementing (and hence, not sharp). In fact, the only complementing (sharp) elements of S are 0 and 1. Moreover, an element of S is nonisotropic if and only if it is greater than 1/2. Hence, although S is not nonisotropic, it is regular. If P is an arbitrary unit SOP, then there exists at least one S-morphism on P. Indeed, define $\phi\colon P \to S$ by $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(a) = 1/2$ for $a \neq 0, 1$. We can make S into an effect algebra with the same order and $'$ as follows. For $a, b \in S$, we say that $a \oplus b$ exists if $a + b \leq 1$ and we then define $a \oplus b = a + b$. The next lemma, which is a special case of a theorem proved in Foulis and Bennett (n.d.), shows that there is only one effect morphism from S to S.

Lemma 3.5. A map $\phi\colon S \to S$ is an effect morphism if and only if $\phi(a) = a$ for all $a \in S$.

Proof. It is clear that $\phi(a) = a$ is an effect morphism. Conversely, suppose $\phi\colon S \to S$ is an effect morphism. For $n \in \mathbb{N}$, since

$$\frac{1}{n} \oplus \cdots \oplus \frac{1}{n} = 1 \qquad (n \text{ summands})$$

we have $n\phi(1/n) = 1$, so $\phi(1/n) = 1/n$. If $m/n \in S$ is rational, then

$$\frac{1}{n} \oplus \cdots \oplus \frac{1}{n} = \frac{m}{n} \qquad (m \text{ summands})$$

Hence, $\phi(m/n) = m\phi(1/n) = m/n$. Let $a \in S$ be irrational. If $r \in S$ is rational and $a < r$, then $\phi(a) \leq \phi(r)$. Hence

$$\begin{aligned}\phi(a) &\leq \inf\{\phi(r)\colon r \text{ rational}, a < r\}\\ &= \inf\{r\colon r \text{ rational}, a < r\} = a\end{aligned}$$

Similarly,

$$\phi(a) \geq \sup\{r\colon r \text{ rational}, r < a\} = a$$

Hence, $\phi(a) = a$. ■

If P is an arbitrary effect algebra, an effect morphism $\phi\colon P \to [0, 1]$ is called a *state*. We conclude that $S = [0, 1]$ has only one state. Of course, this single state is order determining for S. If P and Q are effect algebras, we define their *horizontal sum* $P \oplus Q$ to be the disjoint union of P and Q with their 0's identified and their 1's identified. For $a, b \in P \oplus Q$ we say that $a \oplus b$ is defined if both a, b are in P and $a \oplus_P b$ is defined or both a, b are in Q and $a \oplus_Q b$ is defined. In the first case we define $a \oplus b = a \oplus_P b$ and in the second $a \oplus b = a \oplus_Q b$. It is easy to check that $P \oplus Q$ is an effect algebra. By Lemma 3.5, $S \oplus S$ again has only one state $\phi(a) = a$. But ϕ is no longer order determining, since $\phi(a) = \phi(b)$ if a is in the first S and b is in the second S with $a = b$ as elements of S.

Example 5. Let X be a nonempty set and let $P = [0, 1]^X$. For $f, g \in P$, define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ and define $f' = 1 - f$. Then $(P, 0, 1, \leq, ')$ is a closed, regular SOL that is not complementing (or sharp) in general.

Lemma 3.6. (i) A function $f \in P$ is sharp if and only if $f^2 = f$. (ii) If X is a topological space and P_0 is the sub-SOP of P consisting of continuous functions, then $f \in P_0$ is sharp if and only if f is the characteristic function of a clopen set. In particular, if X is connected, then the only sharp elements of P_0 are 0 and 1.

Proof. (i) It is easy to check that $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$. Hence, the following are equivalent: f is sharp, $\max(f(x), 1 - f(x)) = 1$ for all $x \in X$, for all $x \in X$ we have $f(x) = 1$ or $f(x) = 0$, $f^2 = f$. (ii) Clearly, characteristic functions of clopen sets satisfy $f^2 = f$, so by (i), such functions are sharp. Conversely, if $f \in P_0$ is sharp, then by (i), f is the characteristic function of a set $A \subseteq X$. Since $A = f^{-1}(\{1\})$ and $A^c = f^{-1}(\{0\})$, both A and A^c are closed. Hence, A is clopen. ■

We can make P into an effect algebra with the same order and $'$ as follows. For $f, g \in P$ we say that $f \oplus g$ is defined if $f(x) + g(x) \leq 1$ for all $x \in X$ and we then define $f \oplus g = f + g$. For $x \in X$, define the effect morphism $\phi_x: P \to [0, 1]$ by $\phi_x(f) = f(x)$. Then $\{\phi_x: x \in X\}$ is an order-determining set of states on P. If P and $[0, 1]$ are considered as SOPs, then $\{Q_x: x \in X\}$ is an order-determining set of $[0, 1]$-morphisms on P.

Example 6. Let $B(H)$ be the set of bounded self-adjoint operators on a complex Hilbert space H. For $A, B \in B(H)$, we define $A \leq B$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$ and we define $A' = I - A$. If we let

$$P = \{A \in B(H): 0 \leq A \leq I\}$$

then $(P, 0, I, \leq, ')$ is a closed, regular SOP that is not complementing (or sharp). For example, $A = I/2$ is strongly isotropic and hence A is not complementing or sharp. The next result characterizes the sharp (and complementing) elements of P. Recall that $A \in P$ is a *projection* if $A^2 = A$.

Theorem 3.7. An element $A \in P$ is sharp if and only if A is a projection.

Proof. Suppose A is a projection and $B \in P$ satisfies $A \leq B$, $A' \leq B$. Then for every $x \in AH$ we have

$$\langle x, x\rangle = \langle Ax, x\rangle \leq \langle Bx, x\rangle \leq \langle x, x\rangle$$

Hence, $\langle Bx, x\rangle = \|x\|^2$ for every $x \in AH$ and similarly $\langle By, y\rangle = \|y\|^2$ for every $y \in A'H$. Since A is a projection, if $u \in H$, then there exist $x \in AH$, $y \in A'H$ such that $u = x + y$. Since $x \perp y$, we have $\|u\|^2 = \|x\|^2 + \|y\|^2$. Hence,

$$\begin{aligned}\langle Bu, u\rangle &= \langle Bx + By, x + y\rangle \\ &= \langle Bx, x\rangle + \langle By, y\rangle + \langle Bx, y\rangle + \langle By, x\rangle \\ &= \|x\|^2 + \|y\|^2 + \langle Bx, y\rangle + \langle By, x\rangle \\ &= \|u\|^2 + \langle Bx, y\rangle + \langle By, x\rangle\end{aligned}$$

Since $\langle Bu, u\rangle \leq \|u\|^2$, we conclude that

$$\langle Bx, y\rangle + \langle By, x\rangle \leq 0$$

Letting $v = x - y$ we have

$$\langle Bv, v\rangle = \|v\|^2 - \langle Bx, y\rangle - \langle By, x\rangle$$

Again, since $\langle Bv, v\rangle \leq \|v\|^2$, we conclude

$$\langle Bx, y\rangle + \langle By, x\rangle = 0$$

Hence, $\langle Bu, u\rangle = \|u\|^2$ and it follows that $B = I$. It follows that $A \vee A' = I$, so A is sharp.

Conversely, suppose $A \in P$ is not a projection. By the spectral theorem, we have

$$A = \int_0^1 \lambda P^A(d\lambda), \qquad A' = \int_0^1 (1 - \lambda)P^A(d\lambda)$$

Let f: $[0, 1] \to [0, 1]$ be the continuous function defined by $f(\lambda) = \max(\lambda, 1 - \lambda)$. Define $B \in P$ by

$$B = \int_0^1 f(\lambda)P^A(d\lambda)$$

It follows that $A \leq B$ and $A' \leq B$. Since A is not a projection, its spectrum $\sigma(A) \not\subseteq \{0, 1\}$. Hence, there exists a $\lambda_0 \in \sigma(A)$ with $0 < \lambda_0 < 1$. Then $P^A(\Delta) \neq 0$ for any neighborhood Δ of λ_0. Let $\Delta = (a, b)$, where $0 < a < \lambda_0 < b < 1$. Let $x \in H$ satisfy $x \neq 0$, $P^A(\Delta)x = x$, and hence $P^A(\Delta^c)x = 0$. Then

$$\begin{aligned}\langle Bx, x\rangle &= \int_0^1 f(\lambda)\langle P^A(d\lambda)x, x\rangle = \int_\Delta f(\lambda)\langle P^A(d\lambda)x, x\rangle \\ &\leq \max_{\lambda \in \Delta} f(\lambda)\|x\|^2 < \|x\|^2\end{aligned}$$

It follows that $B \neq I$. Hence, $A \vee A' \neq I$ and A is not sharp. ■

For a different and independent proof of Theorem 3.7, we refer the reader to Dvurečenskij (n.d.).

We can make P into an effect algebra with the same order and $'$ as follows. For $A, B \in P$ we say that $A \oplus B$ is defined if $A + B \in P$ and we then define $A \oplus B = A + B$. For $x \in H$ with $\|x\| = 1$, define the state ϕ_x on P by $\phi_x(A) = \langle Ax, x\rangle$. Then $\{\phi_x: x \in H\}$ is an order-determining set of states on P. However, as in Example 4, it is easy to show that $P \oplus P$ does not have an order-determining set of states.

Example 7. Let $P = \{A \in B(H): 0 \leq A \leq I\}$ as in Example 6. For $A \in P$, let $\ker(A) = \{x \in H: Ax = 0\}$ and let $E_{\ker(A)}$ be the projection onto $\ker(A)$. Define the unary operation $^\sim$: $P \to P$ by $A^\sim = E_{\ker(A)}$. Then $(P, 0, I, \leq, {}^\sim)$ is a complementing SOP. Indeed, suppose $A \leq B$. If $x \in \ker(B)$, then $\langle Ax, x\rangle \leq \langle Bx, x\rangle = 0$. Since $A \geq 0$, there exists a unique $C \geq 0$ such that $A = C^2$. Hence,

$$\|Cx\|^2 = \langle Cx, Cx\rangle = \langle C^2x, x\rangle = \langle Ax, x\rangle = 0$$

so $Cx = 0$ and then $Ax = C^2x = 0$. Therefore, $x \in \ker(A)$, so $\ker(B) \subseteq \ker(A)$. It follows that $B^\sim \leq A^\sim$. To show that $A \leq A^{\sim\sim}$, we have

$$A^{\sim\sim} = E^\sim_{\ker(A)} = E'_{\ker(A)} = A'^\sim$$

Hence, for all $x \in H$ we have

$$\begin{aligned}\langle Ax, x\rangle &= \langle A(A^{\sim\sim} + A^{\sim})x, (A^{\sim\sim} + A^{\sim})x\rangle \\ &= \langle AA^{\sim\sim}x, A^{\sim\sim}x + A^{\sim}x\rangle \\ &= \langle AA^{\sim\sim}x, A^{\sim\sim}x\rangle + \langle A^{\sim\sim}x, AA^{\sim}x\rangle \\ &= \langle AA^{\sim\sim}x, A^{\sim\sim}x\rangle \le \langle A^{\sim\sim}x, A^{\sim\sim}x\rangle \\ &= \langle A^{\sim\sim}x, x\rangle\end{aligned}$$

To show that $A \wedge A^{\sim} = 0$, suppose $B \le A, A^{\sim}$. Then

$$B \le B^{\sim\sim} \le A^{\sim\sim} = A^{\sim\prime}$$

Since $A^{\sim}$ is a projection, applying Theorem 3.7 gives $B \le A^{\sim} \wedge A^{\sim\prime} = 0$. Hence, $B = 0$, so $A \wedge A^{\sim} = 0$.

A similar structure can be obtained from Example 5. Let $P = [0, 1]^X$ and for $f \in P$, let $f^{\sim} = \chi_{\ker(f)}$, where χ denotes the characteristic function. As in the previous paragraph, $(P, 0, 1, \le, {}^{\sim})$ is a complementing SOP. Except in trivial cases, neither this SOP nor the SOP of the previous paragraph is closed. We also have the identity $f^{\sim\sim} = f^{\sim\prime}$ as before.

In general, $(P, 0, 1, \le, {}', {}^{\sim})$ is called a BZ-*poset* if $(P, 0, 1, \le, {}')$ is a closed, regular SOP, $(P, 0, 1, \le, {}^{\sim})$ is a complementing SOP, and $a^{\sim\sim} = a^{\sim\prime}$ for all $a \in P$. For any BZ-poset, we have $a^{\sim} \le a'$. Indeed, since $a \le a^{\sim\sim} = a^{\sim\prime}$, we have $a^{\sim} = a^{\sim\prime\prime} \le a'$. Investigations of BZ-posets can be found in Cattaneo and Nistico (1989), Dalla Chiara and Giuntini (1989), Giuntini (1990), and Giuntini and Greuling (1989).

4. EXTENSIONS OF PARTIAL MORPHISMS

Let P and Q be SOPs. A partial Q-morphism ϕ on P is *maximal* if there are no partial Q-morphisms on P that properly extend ϕ.

Lemma 4.1. If P is a SOP, then any partial Q-morphism on P has a maximal extension.

Proof. Let P_0 be a sub-SOP of P and let ϕ_0 be a partial Q-morphism with domain P_0. Let $\mathcal{M}$ be the set of all partial Q-morphism extensions of ϕ_0. Partially order $\mathcal{M}$ by $\phi_1 \le \phi_2$ if ϕ_2 is an extension of ϕ_1. Let $C = \{\phi_\alpha: \alpha \in A\}$ be a chain in $\mathcal{M}$. Denote the domain of ϕ_α by $D(\phi_\alpha)$ and let $D = \cup\{D(\phi_\alpha): \alpha \in A\}$. Define $\phi: D \to Q$ as follows. If $a \in D$, then $a \in D(\phi_\alpha)$ for some $\alpha \in A$ and we define $\phi(a) = \phi_\alpha(a)$. We first show that ϕ is well defined. Suppose $a \in D(\phi_\alpha) \cap D(\phi_\beta)$. Without loss of generality, we can assume that $\phi_\alpha \le \phi_\beta$. Hence, $\phi_\alpha(a) = \phi_\beta(a)$ and ϕ is well defined. Next, D is a sub-SOP of P since $0, 1 \in D$ and if $a \in D$, then $a \in D(\phi_\alpha)$ for some

$\alpha \in A$, so $a' \in D(\phi_\alpha) \subseteq D$. It is also clear that ϕ is an extension of ϕ_α for all $\alpha \in A$. To show that ϕ is a partial Q-morphism on P, let $a \in D$. Then $a \in D(\phi_\alpha)$ for some $\alpha \in A$, so $a' \in D(\phi_\alpha)$ and

$$\phi(a') = \phi_\alpha(a') = \phi_\alpha(a)' = \phi(a)'$$

Suppose $a, b \in D$ with $a \le b$. Then $a \in D(\phi_\alpha)$ and $b \in D(\phi_\beta)$ for some $\alpha, \beta \in A$. Without loss of generality, assume that $\phi_\beta \le \phi_\alpha$. Then $a, b \in D(\phi_\alpha)$ and

$$\phi(a) = \phi_\alpha(a) \le \phi_\alpha(\beta) = \phi(b)$$

Clearly, $\phi(0) = 0$. Hence, $\phi \in \mathcal{M}$ and ϕ is an upper bound for C. By Zorn's lemma, $\mathcal{M}$ has a maximal element. ■

The next result shows that any SOP has an abundance of [0, 1]-morphisms, where [0, 1] is the SOP defined in Example 4.

Theorem 4.2. If P is a SOP, then any partial [0, 1]-morphism ϕ_0 on P has an extension to a [0, 1]-morphism on P.

Proof. Applying Lemma 4.1, there exists a maximal partial [0, 1]-morphism ϕ that extends ϕ_0. Assume that $D(\phi) \ne P$, and let $a \in P \backslash D(\phi)$. Suppose a' (and hence, a'') is in $D(\phi)$. If $P_1 = D(\phi) \cup \{a\}$, then P_1 is a sub-SOP of P that properly contains $D(\phi)$. Define ϕ_1: $P_1 \to [0, 1]$ by $\phi_1(b) = \phi(b)$ if $b \in D(\phi)$ and $\phi_1(a) = \phi(a'')$. Then ϕ_1 properly extends ϕ and $\phi_1(b') = 1 - \phi_1(b)$ for every $b \in P_1$. To show that ϕ_1 is a partial [0, 1]-morphism, let $b, c \in D(\phi)$ with $b \le c$. Then

$$\phi_1(b) = \phi(b) \le \phi(c) = \phi_1(c)$$

If $b \in D(\phi)$ and $b \le a$, then $a' \le b'$, so

$$\phi_1(a') = \phi(a') \le \phi(b') = \phi_1(b')$$

Hence,

$$\phi_1(b) = 1 - \phi_1(b') \le 1 - \phi_1(a') = \phi_1(a)$$

Similarly, $a \le b$ implies that $\phi_1(a) \le \phi_1(b)$. Hence, ϕ_1 is a partial [0, 1]-morphism that properly extends ϕ. This contradicts the maximality of ϕ. We conclude that $a', a'' \in P \backslash D(\phi)$. Let $P_1 = D(\phi) \cup \{a', a''\}$. Then P_1 is a sub-SOP of P and we define ϕ_1: $P_1 \to [0, 1]$ as follows. If $b \in D(\phi)$, $\phi_1(b) = \phi(b)$. Let

$$\lambda_1 = \sup\{\phi(b): b \in D(\phi), b < a''\}$$

$$\lambda_2 = \inf\{\phi(b): b \in D(\phi), b > a''\}$$

If $b, c \in D(\phi)$ and $b < a'' < c$, then $\phi(b) \leq \phi(c)$. Hence, $\lambda_1 \leq \lambda_2$. If $\lambda_1 \leq 1/2 \leq \lambda_2$, let $\lambda = 1/2$, and otherwise, let $\lambda = \lambda_1$. Notice that $\lambda_1 \leq \lambda \leq \lambda_2$. Define $\phi_1(a'') = \lambda$ and $\phi_1(a') = 1 - \lambda$. Then $\phi_1(c') = 1 - \phi_1(c)$ for all $c \in P_1$. If $c, d \in D(\phi)$ and $c \leq d$, then $\phi_1(c) \leq \phi_1(b)$. Suppose $a' \leq a''$. If $\lambda = 1/2$, then $\phi_1(a'') = \phi_1(a') = 1/2$. If $\lambda > 1/2$, then $\phi_1(a'') = \lambda_1 > 1/2$, so $\phi_1(a') < 1/2$ and $\phi_1(a') < \phi_1(a'')$. Suppose $\lambda < 1/2$. Then there exists a $b \in D(\phi)$ such that $a'' < b$ and $\phi(b) < 1/2$. But $b' \leq a' \leq a'' < b$, so $\phi(b') \leq \phi(b) < 1/2$, which is a contradiction. Suppose $a'' \leq a'$. If $\lambda = 1/2$, we again have $\phi_1(a'') = \phi_1(a') = 1/2$. If $\lambda < 1/2$, then $\phi_1(a'') = \lambda_1 < 1/2$, so $\phi_1(a') > 1/2$. Hence, $\phi_1(a'') < \phi_1(a')$. Suppose $\lambda > 1/2$. Then there exists a $b \in D(\phi)$ such that $b < a''$ and $\phi(b) > 1/2$. But $b' \geq a' \geq a'' > b$, so $\phi(b') \geq \phi(b) > 1/2$, which is a contradiction. Suppose $c \in D(\phi)$ and $c < a''$. Then

$$\phi_1(c) = \phi(c) \leq \lambda_1 \leq \lambda = \phi_1(a'')$$

Suppose $c \in D(\phi)$ and $a'' < c$. Then

$$\phi_1(c) = \phi(c) \geq \lambda_2 \geq \lambda = \phi_1(a'')$$

Suppose $c \in D(\phi)$ and $c < a'$. Then $a'' \leq c'$, so by the above, $\phi_1(a'') \leq \phi_1(c')$. Hence,

$$\phi_1(c) = 1 - \phi_1(c') \leq 1 - \phi_1(a'') = \phi_1(a')$$

Finally, suppose $c \in D(\phi)$ and $a' < c$. Then $c' \leq a''$, so $\phi_1(c') \leq \phi_1(a'')$. Hence,

$$\phi_1(c) = 1 - \phi_1(c') \geq 1 - \phi_1(a'') = \phi_1(a')$$

We concluded that ϕ_1 is a partial $[0, 1]$-morphism that properly extends ϕ. This again contradicts the maximality of ϕ. Hence, $D(\phi) = P$, so ϕ is a $[0, 1]$-morphism on P that extends ϕ_0. ■

To illustrate the utility of the abundance of $[0, 1]$-morphisms, we now characterize various properties in terms of these morphisms. Other results of this type will be given in later sections.

Corollary 4.3. Let P be a SOP and let M be the set of all $[0, 1]$-morphisms on P. (i) $a \in P$ is nonisotropic if and only if there exists an $m \in M$ such that $m(a) = 1$. (ii) $a \in P$ is isotropic if and only if $m(a) \leq 1/2$ for all $m \in M$. (iii) $a \in P$ is strongly isotropic if and only if $m(a) = 1/2$ for all $m \in M$. (iv) P is complementing if and only if for every $a \neq 0$ in P there exists an $m \in M$ such that $m(a) = 1$. (v) P is a unit SOP if and only if for every $a \neq 0$ in P there exists an $m \in M$ such that $m(a) \neq 0$.

Proof. (i) Suppose $a \in P$ is nonisotropic and let P_1 be the sub-SOP given by

$$P_1 = \{0, 1, 1', a, a', a''\}$$

Since $a \leq 1$, we have $1' \leq a'$, and since $a' \leq 1$, we have $1' \leq a''$. If $a \leq 1'$, then $a \leq 1 = 1'' = a'$, which is a contradiction.

Case 1. Assume that a and a' are related. Then $a' \leq a$, so $a' \leq a''$. Define the partial [0, 1]-morphism m_1 with domain P_1 by $m_1(0) = m_1(1') = m_1(a') = 0$, $m_1(1) = m_1(a) = m_1(a'') = 1$. Applying Theorem 4.2, we can extend m_1 to an $m \in M$.

Case 2. Assume that a and a' are unrelated but a' and a'' are related. We cannot have $a'' \leq a'$, since then $a \leq a'' \leq a'$. Hence, $a' \leq a''$ and we proceed as in Case 1.

Case 3. Assume that a, a' and a', a'' are unrelated. Again proceed as in Case 1. If a is isotropic, then $a \leq a'$. Hence, for any $m \in M$ we have $m(a) \leq m(a') = 1 - m(a)$, so $m(a) \leq 1/2$. We conclude that $m(a) \neq 1$ for every $m \in M$. Parts (ii) and (iii) follow from (i). Part (iv) follows from (i) and Lemma 2.2(iv). (v) If P is a unit SOP and $a \neq 0$, then the [0, 1]-morphism ϕ defined in Example 4 satisfies $\phi(a) \neq 0$. Conversely, if P is not a unit SOP, then $1' \neq 0$ and yet $m(1') = 1 - m(1) = 0$ for all $m \in M$. ■

5. CLOSED, REGULAR SOPs

We have seen from Examples 4–6 that there are important closed, regular SOPs. In this section we characterize such SOPs in terms of their [0, 1]-morphism sets and their representations as function spaces. Although the proof of the next result is tedious, we present it in detail because various parts will be needed for subsequent theorems.

Theorem 5.1. If P is a closed, regular SOP, then P has an order-determining set of [0, 1]-morphisms.

Proof. Let M be the set of all [0, 1]-morphisms on P. It suffices to show that if $a, b \in P$ with $a \not\leq b$ then there exists an $m \in M$ such that $m(a) > m(b)$. Assume that $a \not\leq b$ and form the sub-SOP

$$P_1 = \{0, 1, a, b, a', b'\}$$

It suffices to find a partial [0–1]-morphism m with domain P_1 such that $m(a) > m(b)$ because we can then apply Theorem 4.2 to extend m to a [0–1]-morphism on P. In the following we always have $m(0) = 0$, $m(1) = 1$.

Case 1. Assume that a, b, a', b' are nonisotropic.

(i) $b < a$, $b \leq a'$. Then $b < a \leq b'$, contradiction.

(ii) $b < a$, $a' \leq b$. Then $a' \leq b < a$, contradiction.

(iii) $b < a$, b and a' unrelated. Then $a' < b'$. Let $m(a) = m(b') = 1$, $m(b) = m(a') = 0$.

(iv) b and a unrelated, $b \leq a'$. Let m be as in (iii).

(v) b, a unrelated and b, a' unrelated. Let m be as in (iii).

Case 2. Assume that a, b, a' are nonisotropic and $b' \leq b$.
(i) $b < a$, $b \leq a'$. Then $b < a \leq b'$, contradiction.
(ii) $b < a$, $a' \leq b$. Then $a' \leq b < a$, contradiction.
(iii) $b < a$, b and a' unrelated. Then $b' \leq b < a$, contradiction.
(iv) b and a unrelated, $b \leq a'$. Then $a \leq b' \leq b$, contradiction.
(v) b, a unrelated and b, a' unrelated. Let $m(a) = 1$, $m(a') = 0$, $m(b) = m(b') = 1/2$.

Case 3. Assume that a, b, b' are nonisotropic and $a' \leq a$.
(i) $b < a$, $b \leq a'$. Then $b < a \leq b'$, contradiction.
(ii) $b < a$, $a' \leq b$. Let m be as in Case 1(iii).
(iii) $b < a$, b and a' unrelated. Let m be as in (ii).
(iv) b and a unrelated, $b \leq a'$. Then $b \leq a' \leq a$, contradiction.
(v) b, a unrelated and b, a' unrelated. Let m be as in (ii).

Case 4. Assume that a, b', a' are nonisotropic and $b \leq b'$.
(i) $b < a$, $b \leq a'$. Let m be as in Case 1(iii). For (ii)–(v) let m be as in (i).

Case 5. Assume that b, a', b' are nonisotropic and $a \leq a'$.
(i) $b < a$, $b \leq a'$. Then $b \leq a' < b'$, contradiction.
(ii) $b < a$, $a' \leq b$. Then $b < a \leq a' \leq b$, contradiction.
(iii) $b < a$, b and a' unrelated. Then $b < a \leq a'$, contradiction.
(iv) b and a unrelated, $b \leq a'$. Let $m(a) = m(a') = 1/2$, $m(b) = 0$, $m(b') = 1$.
(v) b, a unrelated and b, a' unrelated. Let m be as in (iv).

Case 6. Assume that a, b are nonisotropic and $a' \leq a$, $b' \leq b$. Since P is regular, we have $a' \leq b$.
(i) $b < a$. Let m be as in Case 2(v).
(ii) b and a unrelated. Let m be as in (i).

Case 7. Assume that a', b' are nonisotropic and $a \leq a'$, $b \leq b'$. Since P is regular, we have $a \leq b'$.
(i) $b < a$. Let m be as in Case 5(iv).
(ii) b and a unrelated. Let m be as in (i).

Case 8. Assume that a and a' are nonisotropic and $b = b'$.
(i) $b < a$. Then $b' < a$. Let m be as in Case 2(v).
(ii) b and a unrelated. Then b and a' are unrelated. Let m be as in (i).

Case 9. Assume that b and b' are nonisotropic and $a = a'$.
(i) $b < a$. Then $b < a'$. Let m be as in Case 5(iv).
(ii) b and a unrelated. Then b and a' are unrelated. Let m be as in (i).

Case 10. Assume that a and b' are nonisotropic and $b \leq b'$, $a \leq a'$. Let m be as in Case 1(iii).

Case 11. Assume that a' and b are nonisotropic and $a \leq a'$, $b' \leq b$. Since P is regular, $a \leq b$, which is a contradiction.

Case 12. Assume that a is nonisotropic and b, a', b' are isotropic. Then $b = b'$, $b < a$, and $a' < b$. Let m be as in Case 2(v).

Case 13. Assume that b is nonisotropic and a, a', b' are isotropic. Then $a \leq b$, which is a contradiction.

Case 14. Assume that a' is nonisotropic and a, b, b' are isotropic. Then $a \leq b$, which is a contradiction.

Case 15. Assume that b' is nonisotropic and a, b, a' are isotropic. Let m be as in Case 5(iv).

Case 16. Assume that a, b, a', b' are isotropic. Then $a = b$, which is a contradiction. ■

Let X be a nonempty set and let Q be a SOP. For $f, g \in Q^X$ define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ and define f' by $f'(x) = f(x)'$ for all $x \in X$. Moreover, we define $0, 1 \in Q^X$ by $0(x) = 0$, $1(x) = 1$ for $x \in X$. Let $P \subseteq Q^X$ satisfy $0 \in P$ and $f \in P$ implies $f' \in P$. Then $(P, 0, 1, \leq, ')$ is a SOP that will call a *Q-function* SOP. The [0, 1]-function SOPs are particularly interesting because they correspond to a fuzzy logic or a fuzzy set theory. We shall presently show that the class of closed, regular SOPs coincides with the class of [0, 1]-function SOPs.

It is also of interest to consider the sub-SOP $\{0, 1\} \subseteq [0, 1]$. Of course, $\{0, 1\}$ is the simplest nontrivial SOP. Let P be a $\{0, 1\}$-function SOP. If we identify a set with its characteristic function, then P becomes a set-SOP. Moreover, in this case, if $A \in P$ is considered as a set, then $A' = A^c$. We then call $(P, \emptyset, X, \subseteq, ^c)$ a *standard set*-SOP. We shall also show that the class of orthoposets coincides with the class of standard set-SOPs.

Theorem 5.2. (i) A SOP P has an order-determining set of Q-morphisms if and only if P is isomorphic to a Q-function SOP. (ii) Suppose P has an order-determining set of Q-morphisms. If Q is a closed, regular, complementing, sharp, unit SOP, respectively, then P has these properties, respectively.

Proof. (i) Suppose P has an order-determining set of Q-morphisms M. Define $\phi: P \to Q^M$ by $[\phi(a)](m) = m(a)$ and let $F = \{\phi(a): a \in P\}$. Since

$$[\phi(a')](m) = m(a') = m(a)' = [\phi(a)](m)' = [\phi(a)'](m)$$

for all $m \in M$, we have $\phi(a') = \phi(a)'$. We conclude that $f \in F$ implies $f' \in F$. Since $\phi(0) = 0$, we have $0 \in F$, so $(F, 0, 1, \leq, ')$ is a Q-function SOP.

If $a, b \in P$ with $a \neq b$, since M is order determining, there exists an $m \in M$ such that $m(a) \neq m(b)$. Hence, $[\phi(a)](m) \neq [\phi(b)](m)$, so $\phi(a) \neq \phi(b)$. We conclude that $\phi: P \to F$ is bijective. To show that ϕ is an isomorphism, suppose $a, b \in P$ with $a \leq b$. Then

$$[\phi(a)](m) = m(a) \leq m(a) = [\phi(b)](m)$$

for all $m \in M$, so $\phi(a) \leq \phi(b)$. Finally, if $\phi(a) \leq \phi(b)$, then for any $m \in M$ we have

$$m(a) = [\phi(a)](m) \leq [\phi(b)](m) = m(b)$$

Since M is order determining, we have $a \leq b$. We conclude that P and F are isomorphic. Conversely, suppose there exists an isomorphism $\phi: P \to F$, where $F \subseteq Q^X$ is a Q-function SOP. For $x \in X$, define $m_x: P \to Q$ by $m_x(a) = [\phi(a)](x)$. Then $m_x(0) = [\phi(0)](x) = 0$ and

$$m_x(a') = [\phi(a')](x) = [\phi(a)'](x) = [\phi(a)](x)' = m_x(a)'$$

If $a, b \in P$ with $a \leq b$, then $\phi(a) \leq \phi(b)$ so

$$m_x(a) = [\phi(a)](x) \leq [\phi(b)](x) = m_x(b)$$

Hence, for any $x \in X$, m_x is a Q-morphism on P. Suppose $m_x(a) \leq m_x(b)$ for all $x \in X$. Then

$$[\phi(a)](x) = m_x(a) \leq m_x(b) = [\phi(b)](x)$$

for every $x \in X$, so $\phi(a) \leq \phi(b)$. It follows that $a \leq b$, so $\{m_x: x \in X\}$ is an order-determining set of Q-morphisms on P. (ii) Let M be an order-determining set of Q-morphisms on P. Assume that Q is closed. Then for any $a \in P$, $m \in M$, we have $m(a'') = m(a)'' = m(a)$. Since M is order determining, we have $a'' = a$, so P is closed. Assume that Q is regular. Let $a, b \in P$ with $a \perp a$ and $b \perp b$. Then $m(a) \perp m(a)$ and $m(b) \perp m(b)$ for every $m \in M$. Hence, $m(a) \perp m(b)$, so $m(a) \leq m(b)' = m(b')$ for every $m \in M$. Since M is order determining, $a \leq b'$, so $a \perp b$. Assume that Q is complementing. If $b \leq a, a'$, then $m(b) \leq m(a), m(a)'$. Hence, $m(b) \leq m(a) \wedge m(a)' = 0$, so $m(b) = 0$ for every $m \in M$. Hence, $b = 0$, so P is complementing. The proof that Q is sharp implies P is sharp is similar. Assume that Q is a unit SOP. Then $m(1') = m(1)' = 1' = 0$ for all $m \in M$. Hence, $1' = 0$, so P is a unit SOP. ∎

Theorem 5.3. For a SOP P, the following statements are equivalent. (i) P is closed and regular, (ii) P has an order-determining set of $[0, 1]$-morphisms, (iii) P is isomorphic to a $[0, 1]$-function SOP.

Proof. That (i) implies (ii) is given by Theorem 5.1. That (ii) and (iii) are equivalent is given by Theorem 5.2(i). Since [0, 1] is closed and regular, that (ii) implies (i) is given by Theorem 5.2(ii). ■

Another interesting sub-SOP of [0, 1] is {0, 1/2, 1} and of course this sub-SOP is closed and regular. An examination of the proof of Theorem 4.2 and Theorem 5.1 shows that a closed, regular SOP has an order-determining set of {0, 1/2, 1}-morphisms. This is analogous to a three-valued logic. We thus have the following corollary.

Corollary 5.4. For a SOP P, the following statements are equivalent. (i) P is closed and regular, (ii) P has an order-determining set of {0, 1/2, 1}-morphisms, (iii) P is isomorphic to a {0, 1/2, 1}-function SOP.

We now consider orthoposets. Since an orthoposet is nonisotropic, Case 1 of the proof of Theorem 5.1 and Theorem 5.2 give the following corollary. That (i) implies (iii) in this corollary has been proved by Katrnoška (1982).

Corollary 5.5. For a SOP P, the following statements are equivalent. (i) P is an orthoposet, (ii) P has an order-determining set of {0, 1}-morphisms, and (iii) P is isomorphic to a standard set-SOP.

6. CLOSED SOPs

We have seen in Section 5 that a closed, regular SOP has an order-determining set of [0, 1]-morphisms. Theorem 5.2(ii) shows that this result cannot hold for an arbitrary closed SOP P. However, we can obtain a slightly weaker result if P contains at most one strongly isotropic element. A set of Q-morphisms M on a SOP P is *separating* if $m(a) = m(b)$ for all $m \in M$ implies that $a = b$.

Theorem 6.1. For a SOP P, the following statements are equivalent. (i) P is closed and contains at most one strongly isotropic element, (ii) P has a separating set of [0, 1]-morphisms, (iii) there is a bijective morphism from P onto a [0, 1]-function SOP.

Proof. Suppose (i) holds and let M be the set of all [0, 1]-morphisms on P. To prove (ii), it suffices to show that if $a, b \in P$ with $a \neq b$, then there exists an $m \in M$ such that $m(a) \neq m(b)$. We can assume without loss of generality that $a \nleq b$. Proceeding as in the proof of Theorem 5.1, we form the sub-SOP

$$P_1 = \{0, 1, a, b, a', b'\}$$

As before, it suffices to find a partial [0, 1]-morphism m with domain P_1 such that $m(a) \neq m(b)$. Cases 1–10, 12, 15, and 16 in the proof of Theorem

5.1 apply unchanged. In Case 11, we let $m(a) = m(b') = 0$ and $m(a') = m(b) = 1$. In Case 13, we let $m(a) = m(a') = 1/2$, $m(b') = 0$, $m(b) = 1$. In Case 14, we let $m(b) = m(b') = 1/2$, $m(a) = 0$, $m(a') = 1$. That (ii) and (iii) are equivalent is similar to the proof of Theorem 5.2(i). Finally, suppose (ii) holds and M is a separating set of $[0, 1]$-morphisms on P. If $a, b \in P$ are strongly isotropic, then $m(a) = m(b) = 1/2$ for every $m \in M$. Hence, $a = b$. ■

Letting $S_0 = \{0, 1\}$, $S_1 = \{0, 1/2, 1\}$, we have seen that an orthoposet has an order-determining set of S_0-morphisms and a closed, regular SOP has an order-determining set of S_1-morphisms. Moreover, it is clear that S_0 and S_1 are the simplest SOPs with these properties. We now seek a SOP S_2 that has this property for an arbitrary closed SOP. By Theorem 5.2(ii), S_2 must be closed, but not regular. The simplest such SOP is $S_2 = \{0, 1, \alpha, \beta\}$, where $\alpha = \alpha'$, $\beta = \beta'$ (α and β are strongly isotropic), and α, β are unrelated. Notice that S_2 is the horizontal sum $S_2 = S_1 \oplus S_1$.

Theorem 6.2. If P is a closed SOP, then any partial S_2-morphism ϕ_0 on P has an extension to an S_2-morphism on P.

Proof. Applying Lemma 4.1, there exists a maximal partial S_2-morphism ϕ that extends ϕ_0. Assume that $D(\phi) \neq P$ and let $a \in P \backslash D(\phi)$. Then $a' \in P \backslash D(\phi)$ and we form the sub-SOP $P_1 = D(\phi) \cup \{a, a'\}$. Define $\phi_1: P_1 \to S_2$ as follows. If $b \in D(\phi)$, $\phi_1(b) = \phi(b)$. Noticing that S_2 is a lattice, let

$$\lambda_1 = \vee\{\phi(b): b \in D(\phi), b < a\}$$

$$\lambda_2 = \wedge\{\phi(b): b \in D(\phi), b > a\}$$

We then have $\lambda_1 \leq \lambda_2$. Define

$$\phi_1(a) = \begin{cases} 0 & \text{if } \lambda_1 = \lambda_2 = 0 \\ 1 & \text{if } \lambda_1 = \lambda_2 = 1 \\ \alpha & \text{if } \lambda_1 = 0, \lambda_2 = 1 \quad \text{or if } \quad \lambda_1 \text{ or } \lambda_2 = \alpha \\ \beta & \text{if } \lambda_1 \text{ or } \lambda_2 = \beta \end{cases}$$

and define $\phi_1(a') = \phi_1(a)'$. Notice that $\lambda_1 \leq \phi_1(a) \leq \lambda_2$ and $\phi_1(c') = \phi_1(c)'$ for all $c \in P_1$. If $b, c \in D(\phi)$ with $b \leq c$, then $\phi_1(b) \leq \phi_1(c)$. Suppose that $a \leq a'$. Then $\phi_1(a) \leq \phi_1(a')$ unless $\phi_1(a) = 1$. But then $\lambda_1 = 1$, so there exists a $b \in D(\phi)$ such that $b < a$ and $\phi(b) = 1$ or there exists $b, c \in D(\phi)$ such that $b, c < a$ and $\phi(b) = \alpha$, $\phi(c) = \beta$ (or vice versa). In either case we have $a \leq a' < b'$. Since $\lambda_2 = 1$, we conclude that $\phi(b') = 1$, which is a contradiction. Suppose that $a' \leq a$. Then $\phi_1(a') \leq \phi_1(a)$ unless $\phi_1(a) = 0$. But then $\lambda_2 = 0$, so there exists a $b \in D(\phi)$ such that $a < b$ and $\phi(b) = 0$ or there exist $b, c \in D(\phi)$ such that $a < b, c$ and $\phi(b) = \alpha$, $\phi(c) = \beta$ (or vice versa). In either case $b' < a' \leq a$. Since $\lambda_1 = 0$, we have $\phi(b') = 0$, which is a contradiction. Next suppose that $b \in D(\phi)$ with $b < a$. Then $\phi(b) \leq \lambda_1 \leq \phi_1(a)$. If $b \in D(\phi)$ with $a < b$, then $\phi(b) \geq \lambda_2 \geq \phi_1(a)$. Suppose $b \in D(\phi)$ with $b < a'$. Then $a < b'$, so $\phi_1(a) \leq \phi(b') = \phi(b)'$. Hence, $\phi(b) \leq \phi_1(a)' = \phi_1(a')$. Finally, suppose $b \in D(\phi)$ with $a' < b$. Then $b' < a$, so $\phi(b)' = \phi(b') \leq \phi_1(a)$. Hence, $\phi_1(a') = \phi_1(a)' \leq \phi(b)$. We conclude that ϕ_1 is a partial S_2-morphism on P that properly extends ϕ. This contradicts the maximality of ϕ. Hence $D(\phi) = P$. ■

Corollary 6.3. For a SOP P, the following statements are equivalent. (i) P is closed, (ii) P has an order-determining set of S_2-morphisms, and (iii) P is isomorphic to an S_2-function SOP.

Proof. Suppose (i) holds and let M be the set of all S_2-morphisms on P. We proceed as in the proof of Theorem 5.1, except we now apply Theorem 6.2 and show that if $a \not\leq b$, then there exists an $m \in M$ such that $m(a) \not\leq m(b)$. Cases 1–10, 12, and 15 in the proof of Theorem 5.1 apply by replacing 1/2 with α. In the other cases, we let $m(a) = m(a') = \alpha$, $m(b) = m(b') = \beta$. We conclude that (i) implies (ii). That (ii) and (iii) are equivalent follows from Theorem 5.2(i). That (ii) implies (i) follows from Theorem 5.2(ii). ■

As in Example 4 of Section 3, let W be the horizontal sum $W = [0, 1] \oplus [0, 1]$. Then W is an effect algebra and W can also be considered as a closed SOP with the same order and $'$. Since S_2 is a sub-SOP (or sub-effect algebra) of W, we know that any closed SOP has an order-determining set of W-morphisms. Moreover, Corollary 6.3 holds with S_2 replaced by W. We have introduced W because W-valued effect morphisms are more useful than S_2-valued effect morphisms. The reason for this is that W contains long strings of sums, while S_2 does not. Our next theorem shows that any closed SOP admits a canonical imbedding into an effect algebra. Let L_1 be a subset of an effect algebra L. We say that L_1 *generates* L if the smallest sub-effect algebra of L that contains L_1 is L itself. We shall also need the following result. If L is an effect algebra and X is a nonempty set, we can organize L^X into an effect algebra as follows. For $f, g \in L^X$, we say that $f \oplus g$ exists if $f(x) \oplus g(x)$ exists for all $x \in X$ and we then define $f \oplus g(x) = f(x) \oplus g(x)$. Define $0(x) = 0$ and $1(x) = 1$ for all $x \in X$ and $f'(x) = f(x)'$ for all $x \in X$. It is easy to show that $(L^X, 0, 1, \oplus)$ is now an effect algebra with order $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Theorem 6.4. Let P be a closed SOP and let M be an order-determining set of W-morphisms on P. Then there exists an effect algebra Q, an order-determining set of W-valued effect morphisms $\psi(M)$ on Q, a SOP isomorphism $\phi: P \rightarrow \phi(P) \subseteq Q$ such that $\phi(P)$ generates Q, and a bijection $\psi: M \rightarrow \psi(M)$ such that $\psi(m)(\phi(a)) = m(a)$ for every $a \in P$, $m \in M$. If Q_1 is an effect algebra with maps ϕ_1: $P \to Q_1$, ψ_1: $M \to \psi_1(M)$ satisfying the previous conditions, then Q_1 and Q are isomorphic. Moreover, if $a, b \in P$ with $a \perp b$ and there exists a $c \in P$ such that $m(c) = m(a) \oplus m(b)$ for all $m \in M$, then $\phi(c) = \phi(a) \oplus \phi(b)$.

Proof. Applying Corollary 6.3 and the proof of Theorem 5.2(i), we have that the map ϕ: $P \to W^M$ given by $\phi(a)(m) = m(a)$ is a SOP isomorphism onto its range $\phi(P)$. Now W^M is an effect algebra and we define Q to be the sub-effect algebra of W^M generated by $\phi(P)$. For $m \in M$ and $f \in Q$ we define $\psi(m)(f) = f(m) \in W$. Then for any $a \in P$, $m \in M$, we have

$$\psi(m)(\phi(a)) = \phi(a)(m) = m(a)$$

Now $\psi(m)$ is a W-valued effect morphism on Q since $\psi(m)(0) = 0(m) = 0$ and if $f \oplus g$ exists, $f, g \in Q$, then

$$\psi(m)(f \oplus g) = (f \oplus g)(m) = f(m) \oplus g(m) = \psi(m)(f) \oplus \psi(m)(g)$$

To show that ψ: $M \to \psi(M)$ is bijective, suppose $\psi(m_1) = \psi(m_2)$. Then for any $a \in P$ we have

$$m_1(a) = \psi(m_1)(\phi(a)) = \psi(m_2)(\phi(a)) = m_2(a)$$

Hence, $m_1 = m_2$. To show that $\psi(M)$ is order determining on Q, suppose $f, g \in Q$ satisfy $\psi(m)(f) \le \psi(m)(g)$ for every $m \in M$. Then $f(m) \le g(m)$ for every $m \in M$, so $f \le g$.

Let Q_1 be an effect algebra with maps ϕ_1: $P \to Q_1$, ψ_1: $M \to \psi_1(M)$ satisfying the given conditions. Define the map β: $Q_1 \to W^{\psi_1(M)}$ by $\beta(q)(\psi_1(m)) = \psi_1(m)(q)$. It is easy to show that β is an effect isomorphism from Q_1 onto $\beta(Q_1)$. Define α: $W^M \to W^{\psi_1(M)}$ by $\alpha(f)(\psi_1(m)) = f(m)$. Again, it is easy to show that α is an effect isomorphism. For any $a \in P$, $m \in M$, we have

$$\begin{aligned}\beta[\phi_1(a)](\psi_1(m)) &= \psi_1(m)[\phi_1(a)] = m(a) = \phi(a)(m) \\ &= \alpha[\phi(a)](\psi_1(m))\end{aligned}$$

Hence, $\alpha[\phi(P)] = \beta[\phi_1(P)]$, so $\phi(P) = \alpha^{-1} \circ \beta[\phi_1(P)]$. Since $\phi_1(P) \subseteq Q_1$, we have $\phi(P) \subseteq \alpha^{-1} \circ \beta(Q_1)$. The fact that $\phi(P)$ generates Q now gives that $Q \subseteq \alpha^{-1} \circ \beta(Q_1)$. Similarly, $\beta[Q_1(P)] \subseteq \alpha(Q)$ and since $\phi_1(P)$ generates Q_1, we have $\beta[Q_1] \subseteq \alpha(Q)$. Hence, $Q = \alpha^{-1} \circ \beta(Q_1)$ and $\alpha^{-1} \circ \beta$ is an effect isomorphism from Q_1 to Q.

Finally, suppose $a, b \in P$ satisfy the last statement of the theorem. Then for every $m \in M$, we have

$$\begin{aligned}\phi(c)(m) &= m(c) = m(a) \oplus m(b) = \phi(a)(m) \oplus \phi(b)(m) \\ &= [\phi(a) \oplus \phi(b)](m)\end{aligned}$$

Hence, $\phi(c) = \phi(a) \oplus \phi(b)$. ∎

Let (P, M) be a pair where P is a closed SOP and M is an order-determining set of W-morphisms on P. Then Theorem 6.4 states that there exists a unique (up to isomorphism) effect algebra Q that has an order-determining set of W-valued effect morphisms into which P can be canonically imbedded. Moreover, if there are elements of P that "want" to have sums, then the imbedding preserves such sums. In particular, if P is already an effect algebra and the elements of M are effect morphisms, then Q merely reproduces P.

If P is a closed, regular SOP and M is an order-determining set of $[0, 1]$-morphisms on P, then $\psi(M)$ in Theorem 6.4 becomes an order-determining set of states on Q. If P is an orthoposet and M is an order-determining set of $\{0, 1\}$-morphisms on P, then Theorem 6.4 gives the following strong statement. The orthoposet P can be imbedded into an orthoalgebra of sets $(Q, \varnothing, X, \oplus)$ where $A \oplus B$ is defined if and only if A and B are disjoint and in this case $A \oplus B = A \cup B$.

7. SHARP, UNIT SOPs

Although closed or complementing SOPs must be unit SOPs, this does not hold for sharp SOPs. For example, let $P = \{0, 1', 1\}$, where $0 < 1' < 1$. Then P is not a unit SOP, but P is sharp.

In this section, we assume that P is a sharp, unit SOP. We require the unit condition in order to avoid tedious complications. This condition also has the advantage of making P complementing. Continuing our previous program, we introduce the simplest sharp, unit SOP that is not closed. This is the SOP $S_3 = \{0, \alpha, \alpha', \alpha'', 1\}$, where $0 < \alpha < \alpha'' < 1$, $0 < \alpha' < 1$, and there are no other relations. As usual we obtain the following results.

Theorem 7.1. If P is a sharp, unit SOP, then any partial S_3-morphism ϕ_0 on P has an extension to an S_3-morphism ϕ on P.

Proof. As before, there exists a maximal, partial S_3-morphism ϕ on P that extends ϕ_0. Suppose $D(\phi) \neq P$ and let $a \in P\backslash D(\phi)$. Suppose a' (and hence a'') is in $D(\phi)$. Then $P_1 = D(\phi) \cup \{a\}$ is a sub-SOP of P. Define ϕ_1: $P_1 \to S_3$ by $\phi_1(b) = \phi(b)$ if $b \in D(\phi)$ and define

$$\phi_1(a) = \begin{cases} 1 & \text{if } \phi(a') = 0; \qquad 0 \quad \text{if } \phi(a') = 1 \\ \alpha & \text{if } \phi(a') = \alpha' \text{ and } \vee\{\phi(b) : b \in D(\phi), b \leq a\} \leq \alpha \\ \alpha'' & \text{if } \phi(a') = \alpha' \text{ and } \vee\{\phi(b) : b \in D(\phi), b \leq a\} \not\leq \alpha \\ \alpha' & \text{if } \phi(a') = \alpha'' \end{cases}$$

Notice that $\phi(b) \neq \alpha$ if $b \in D(\phi)$ and b is closed. Indeed, otherwise

$$\alpha = \phi(b) = \phi(b'') = \phi(b)'' = \alpha''$$

which is a contradiction. Also, if $b \in D(\phi)$ with $b < a$, then $a' \leq b'$, so $\phi(a') \leq \phi(b')$. Hence, if $\phi(a') = \alpha'$ or 1, then $\phi(b') = \alpha'$ or 1, so $\phi(b) = 0$, α, or α''. Therefore,

$$\vee\{\phi(b): b \in D(\phi), b \leq a\} = 0, \alpha, \text{ or } \alpha''$$

Now ϕ_1 extends ϕ and $\phi_1(b') = \phi_1(b)$ for every $b \in P_1$. If $b, c \in D(\phi)$ with $b \leq c$, then $\phi_1(b) \leq \phi_1(c)$. If $b \in D(\phi)$ with $b < a$, then $a' \leq b'$, so $\phi(a') \leq \phi(b')$. If $\phi(a') = 0$, then $\phi_1(a) = 1$, so $\phi_1(b) \leq \phi_1(a)$. If $\phi(a') = 1$, then $\phi(b)' = \phi(b') = 1$, so $\phi(b) = 0$. Hence, $\phi_1(b) \leq \phi_1(a)$. If $\phi(a') = \alpha'$, then $\phi(b') = \alpha'$ or 1. If $\phi(b') = \alpha'$, then $\phi(b) = \alpha$ or α''. If $\phi(b) = \alpha$, then $\phi_1(b) \leq \phi_1(a)$ because $\phi_1(a) = \alpha$ or α''. If $\phi(b) = \alpha''$, then $\phi_1(b) = \phi_1(a)$. If $\phi(b') = 1$, then $\phi_1(b) = 0 \leq \phi_1(a)$. If $\phi(a') = \alpha''$, then $\phi(b') = \alpha''$ or 1, so $\phi(b) = \alpha'$ or 0. We again have $\phi_1(b) \leq \phi_1(a)$. Next suppose that $a < b$. Then $b' \leq a'$, so $\phi(b') \leq \phi(a')$. If $\phi(a') = \alpha'$, then $\phi(b') = \alpha'$ or 0. If $\phi(b') = \alpha'$, then $\phi(b) = \alpha$ or α''. In the case of $\phi(b) = \alpha''$, we have $\phi_1(a) \leq \phi_1(b)$ since $\phi_1(a) = \alpha$ or α''. Suppose $\phi(b) = \alpha$. If $\phi_1(a) = \alpha$, then $\phi_1(a) \leq \phi_1(b)$. If $\phi_1(a) = \alpha''$, then there exists a $c \in D(\phi)$ with $c < a$ and $\phi(c) = \alpha''$. Then $c < a < b$, so $\phi(b) \geq \alpha''$, which is a contradiction. If $\phi(a') = \alpha''$, then $\phi(b') = 0$ or α''. Hence, $\phi(b) = 1$ or α'. In either case $\phi_1(a) \leq \phi_1(b)$. We conclude that ϕ_1 is a partial S_3-morphism that properly extends ϕ, which is a contradiction.

We then have $a, a', a'' \in P\backslash D(\phi)$. Form the sub-SOP $P_1 = D(\phi) \cup \{a', a''\}$. Define ϕ_1: $P_1 \to S_3$ by $\phi_1(b) = \phi(b)$ if $b \in D(\phi)$, $\phi_1(a') = 1$ if there exists a $b \in D(\phi)$ with $a'' < b$ and $\phi(b) = \alpha$ and otherwise

$$\phi_1(a') = \vee\{\phi(b'): b' \in D(\phi), b' < a'\} \tag{7.1}$$

Finally, define $\phi_1(a'') = \phi_1(a')'$. Then $\phi_1(b') = \phi_1(b)'$ for every $b \in P_1$. Suppose that $b \in D(\phi)$ with $b < a'$. Then $b'' < a'$, so $\phi(b) \leq \phi(b'') \leq \phi_1(a')$. Suppose that $b \in D(\phi)$ with $a' < b$. If $c' \in D(\phi)$ with $c' < a'$, then $c' < b$. Hence, $\phi(c') \leq \phi(b)$. Hence, if $\phi_1(a')$ is given by (7.1), then

$$\phi(b) \geq \vee\{\phi(c'): c' \in D(\phi), c' < a'\} = \phi_1(a')$$

Otherwise, there exists a $c \in D(\phi)$ with $a'' < c$ and $\phi(c) = \alpha$, in which case $\phi_1(a') = 1$. But then $b' \leq a'' < c$, so $\phi(b') \leq \phi(c) = \alpha$. Hence, $\phi(b') = 0$, so $\phi(b) = 1$ and again $\phi_1(a') \leq \phi(b)$. Now suppose that $b \in D(\phi)$ with $b < a''$. Then $a' \leq b'$, so $\phi_1(a') \leq \phi(b')$. Hence, $\phi(b) \leq \phi(b'') \leq \phi_1(a'')$. Finally, suppose that $b \in D(\phi)$ with $a'' < b$. Since $b' \leq a'$, we have $\phi(b') \leq \phi_1(a')$. Suppose that $\phi_1(a') = \alpha'$. Then $\phi(b') = 0$ or α', so $\phi(b) = 1$, α, or α''. If $\phi(b) = 1$ or α'', then $\phi_1(a'') = \alpha'' \leq \phi(b)$. If $\phi(b) = \alpha$, then $\phi_1(a') = 1$, which is a contradiction. Suppose that $\phi_1(a') = \alpha''$. Then $\phi(b') = 0$ or α'', so $\phi(b) = 1$ or α'. In either case, $\phi_1(a'') = \alpha' \leq \phi(b)$. If $\phi_1(a') = 1$, then $\phi_1(a'') = 0 \leq \phi(b)$. If $\phi_1(a') = 0$, then $\phi(b') = 0$. Hence, $\phi(b) =$

1 and $\phi_1(a'') \leq \phi(b)$. We conclude that ϕ_1 is a partial S_3-morphism that properly extends ϕ, which is a contradiction. Therefore, $D(\phi) = P$ and ϕ is an S_3-morphism on P. ■

Theorem 7.2. If P is a sharp, unit SOP, then P has an order-determining set of S_3-morphisms.

Lest we try the reader's patience, we omit the somewhat tedious proof. By now the proof of the following theorem is standard.

Theorem 7.3. For a SOP P, the following statements are equivalent. (i) P is a sharp, unit SOP, (ii) P has an order-determining set of S_3-morphisms, and (iii) P is isomorphic to an S_3-function SOP.

We close this section by mentioning that this program can be carried on to include complementing SOPs. For these we introduce a simple complementing SOP that is not sharp. This is the SOP $S_4 = \{0, 1, \alpha, \omega, \omega'\}$, where

$$0 = \alpha' < \omega' < \alpha < \alpha'' = 1, \qquad 0 < \omega = \omega'' < \alpha < 1$$

and there are no other relations. The counterparts of Theorems 7.1–7.3 hold for complementing SOPs with S_3 replaced by S_4. Of course, one might like to carry this program to the extreme and prove such theorems for an arbitrary SOP, but this appears to be quite difficult.

8. BASIC TENSOR PRODUCTS

This section introduces the concept of tensor products of SOPs, presents some examples, and constructs various basic tensor products.

If P, Q, and R are SOPs, a *bimorphism* $\alpha: P \times Q \to R$ satisfies: (1) if $a, b \in P$, $c, d \in Q$, then $a \leq b$ implies that $\alpha(a, c) \leq \alpha(b, c)$ and $c \leq d$ implies that $\alpha(a, c) \leq \alpha(a, d)$; (2) $\alpha(a, c) \perp \alpha(a', d)$ for all $c, d \in Q$ and $\alpha(a, c) \perp \alpha(b, c')$ for all $a, b \in P$; (3) $\alpha(a, 1)' \leq \alpha(a', 1)$ and $\alpha(1, b)' \leq \alpha(1, b')$ for all $a \in P$, $b \in Q$.

Lemma 8.1. Let $\alpha: P \times Q \to R$ be a bimorphism. (i) If $a, b \in P$ and $c, d \in Q$ with $a \leq b$, and $c \leq d$, then $\alpha(a, c) \leq \alpha(b, d)$. (ii) $\alpha(a, 1)' = \alpha(a', 1)$ and $\alpha(1, b)' = \alpha(1, b')$ for all $a \in P$, $b \in Q$. (iii) If $a \perp b$, then $\alpha(a, c) \perp \alpha(b, d)$ for all $c, d \in Q$ and if $c \perp d$, then $\alpha(a, c) \perp \alpha(b, d)$ for all $a, b \in P$. (iv) If R is a unit SOP, then $\alpha(a, 0) = \alpha(0, b) = 0$ for all $a, b \in P$. (v) If R is a unit SOP, then $\alpha(\cdot, 1)$ and $\alpha(1, \cdot)$ are morphisms from P and Q, respectively, into R.

Proof. (i) Applying (1), we have

$$\alpha(a, c) \leq \alpha(b, c) \leq \alpha(b, d)$$

(ii) Applying (2), we have $\alpha(a', 1) \leq \alpha(a, 1)'$ and the result follows from (3). (iii) If $a \perp b$, then $a \leq b'$, so by (1) and (2) we have

$$\alpha(a, c) \leq \alpha(b', c) \leq \alpha(b, d)'$$

Hence $\alpha(a, c) \perp \alpha(b, d)$. (iv) Applying (i) and (ii), we have

$$\alpha(0, 1)' = \alpha(0', 1) = \alpha(1, 1) = 1$$

Since R is a unit SOP, we obtain

$$\alpha(0, 1) \leq \alpha(0, 1)'' = 1' = 0$$

Hence, $\alpha(0, 1) = 0$, so by (1) we have $\alpha(0, b) \leq \alpha(0, 1) = 0$. Thus $\alpha(0, b) = 0$ and similarly $\alpha(a, 0) = 0$. (v) This follows from (1), (ii), and (iv). ■

Notice that condition (2) is stronger than the condition $\alpha(a, c) \perp \alpha(a', c)$ that one might expect. Moreover, (2) is apparently stronger than the analogous condition for bimorphisms of orthoalgebras and effect algebras (Dvurečenskij, 1995; Foulis and Bennett, 1993). We have three replies to this criticism. First, one of the main motivations for studying these structures is to describe physical systems and (2) can be justified on physical grounds. Second, it is not hard to show that the counterpart to (2) holds in orthoalgebras and effect algebras. Third, (2) automatically holds in many natural examples.

Example 1. Define α: $[0, 1] \times [0, 1] \to [0, 1]$ by $\alpha(a, b) = ab$. It is clear that (1) and (3) hold. To prove (2), for any $a, c, d \in [0, 1]$ we have

$$a(c - d) \leq a(1 - d) \leq 1 - d$$

Hence,

$$ac \leq 1 - d + ad = 1 - (1 - a)d = (a'd)'$$

so $\alpha(a, c) \perp \alpha(a', d)$. An argument similar to the proof of Lemma 3.5 shows that α is the only effect bimorphism from $[0, 1] \times [0, 1]$ to $[0, 1]$.

Example 2. Let P be a SOL and define α: $P \times P \to P$ by $\alpha(a, b) = a \wedge b$. Again, it is clear that (1) and (3) hold. To prove (2) we have

$$a \wedge c \leq a \leq a'' \leq a'' \vee d' \leq (a' \wedge d)'$$

Hence, $\alpha(a, c) \perp \alpha(a', d)$.

Example 3. Let $P = 2^X$, $Q = 2^Y$, $R = 2^{X \times Y}$ be standard set-SOPs and define α: $P \times Q \to R$ by $\alpha(A, B) = A \times B$, where $A \times B$ is the usual set-theoretic Cartesian product. It is clear that (1)–(3) hold.

Example 4. Let $P = [0, 1]^X$, $Q = [0, 1]^Y$, $R = [0, 1]^{X \times Y}$ be $[0, 1]$-function SOPs and define α: $P \times Q \to R$ by $\alpha(f, g) = f \otimes g$, where $f \otimes$

$g(x, y) = f(x)g(y)$. Then (1) and (3) clearly hold and (2) follows from Example 1.

Example 5. As in Example 6 of Section 3, let $P = \{A \in B(H_1): 0 \leq A \leq I\}$, $Q = \{A \in B(H_2): 0 \leq A \leq I\}$, $R = \{A \in B(H_1 \otimes H_2): 0 \leq A \leq I\}$, where $H_1 \otimes H_2$ is the usual tensor product and define $\alpha: P \times Q \to R$ by $\alpha(A, B) = A \otimes B$. It is easy to verify (1) and (3). To verify (2), let $A \in P$ and $C, D \in Q$. We must show that

$$A \otimes C \leq I \otimes I - (I - A) \otimes D$$

This is equivalent to showing that

$$\langle Ax, x\rangle\langle Cy, y\rangle \leq \|x\|^2\|y\|^2 - \|x\|^2\langle Dy, y\rangle + \langle Ax, x\rangle\langle Dy, y\rangle$$

for all $x \in H_1$, $y \in H_2$. By Example 1, this inequality holds for all unit vector $x \in H_1$, $y \in H_2$. It then follows that the inequality holds for all $x \in H_1$, $y \in H_2$.

We say that two SOPs P and Q are *of the same type* if P and Q are both closed, regular, sharp, or complementing, respectively. Let P and Q be SOPs of the same type. A *tensor product* of P and Q is a pair (T, τ) where T is a SOP of this type and $\tau: P \times Q \to T$ is a bimorphism satisfying (1) if $\alpha: P \times Q \to R$ is a bimorphism, where R is of this type, then there exists a morphism $\phi: T \to R$ such that $\alpha = \phi \circ \tau$, and (2) T is generated by $\tau(P \times Q)$.

Lemma 8.2. If (T, τ) and (T^*, τ^*) are tensor products of P and Q, then there exists a unique isomorphism $\phi: T \to T^*$ such that $\tau^* = \phi \circ \tau$.

Proof. Since (T, τ) and (T^*, τ^*) are tensor products, there exist morphisms $\phi: T \to T^*$ such that $\tau^* = \phi \circ \tau$ and $\phi^*: T^* \to T$ such that $\tau = \phi^* \circ \tau^*$. Since $\tau(P \times Q)$ and $\tau^*(P \times Q)$ generate T and T^*, respectively, we have

$$T = \{\tau(a, b), \tau(a, b)', \tau(a, b)'': (a, b) \in P \times Q\}$$

$$T^* = \{\tau^*(a, b), \tau^*(a, b)', \tau^*(a, b)'': (a, b) \in P \times Q\}$$

Now for $(a, b) \in P \times Q$, we have

$$\phi^*[\phi(\tau(a, b))] = \phi^*[\tau^*(a, b)] = \tau(a, b)$$

$$\phi[\phi^*(\tau^*(a, b))] = \phi[\tau(a, b)] = \tau^*(a, b)$$

Moreover,

$$\begin{aligned}\phi^*[\phi(\tau(a, b)')] &= \phi^*[\phi(\tau^*(a, b))'] \\ &= \phi^*[\phi(\tau(a, b))]' = \tau(a, b)'\end{aligned}$$

Similarly, $\phi[\phi^*(\tau(a, b)')] = \tau(a, b)'$, $\phi^*[\phi(\tau(a, b)'')] = \tau(a, b)''$, and $\phi[\phi^*(\tau(a, b)'')] = \tau(a, b)''$. We conclude that $\phi^* = \phi^{-1}$, so ϕ is an isomorphism. Clearly, ϕ is unique. ∎

Lemma 8.2 states that if the tensor product of two SOPs exists, it is unique to within an isomorphism. We now construct some basic tensor products. First, consider the orthoposet $S_0 = \{0, 1\}$.

Lemma 8.3. The tensor product of S_0 and S_0 is (S_0, τ_0), where $\tau_0(a, b) = ab$ for all $a, b \in S_0$.

Proof. It is clear that τ_0 is a bimorphism. Let $\alpha\colon S_0 \times S_0 \to R$ be a bimorphism, where R is an orthoposet. Applying Lemma 8.1, we have

$$\alpha(0, 0) = \alpha(0, 1) = \alpha(1, 0) = 0$$

and $\alpha(1, 1) = 1$. Define $\phi\colon S_0 \to R$ by $\phi(0) = 0$, $\phi(1) = 1$. Then clearly, ϕ is a morphism and $\alpha = \phi \circ \tau_0$. Finally, it is trivial that S_0 is generated by $\tau(S_0 \times S_0)$. ∎

Next, consider the closed, regular SOP $S_1 = \{0, 1/2, 1\}$. Define the SOP

$$S_1 \otimes S_1 = \{0, 1/4, 1/2, 3/4, 1\}$$

where $S_1 \otimes S_1$ is considered as a sub-SOP of $[0, 1]$. Then $S_1 \otimes S_1$ is a closed, regular SOP.

Theorem 8.4. The tensor product of S_1 and S_1 is $(S_1 \otimes S_1, \tau_1)$, where $\tau_1(a, b) = ab$ for all $a, b \in S_1$.

Proof. As in Example 1, $\tau_1\colon S_1 \times S_1 \to S_1 \otimes S_1$ is a bimorphism. Since $(1/4)' = 3/4$, $S_1 \otimes S_1$ is generated by $\tau_1(S_1 \times S_1)$. Let $\alpha\colon S_1 \times S_1 \to R$ be a bimorphism, where R is a closed, regular SOP. We then have

$$\alpha(\tfrac{1}{2}, 1) = \alpha(\tfrac{1}{2}', 1) = \alpha(\tfrac{1}{2}, 1)'$$

and similarly $\alpha(1, 1/2) = \alpha(1, 1/2)'$. Since R is regular, we have $\alpha(1/2, 1) = \alpha(1, 1/2)$. Define $\phi\colon S_1 \otimes S_1 \to R$ by $\phi(0) = \alpha(0, 0)$, $\phi(1/4) = \alpha(1/2, 1/2)$, $\phi(1/2) = \alpha(1/2, 1)$, $\phi(3/4) = \alpha(1/2, 1/2)'$, $\phi(1) = \alpha(1, 1)$. Then $\alpha = \phi \circ \tau_1$ and ϕ preserves $'$. It is clear that if $a \le b$, then $\phi(a) \le \phi(b)$ for $a, b \in \{0, 1/4, 1/2, 1\}$. Also, $\phi(0) \le \phi(3/4) \le \phi(1)$. Moreover, since $1/2 \perp 1/2$, we have $\phi(1/4), \phi(1/2) \le \phi(3/4)$. Hence, ϕ is a morphism. ∎

We now come to the closed SOP $S_2 = \{0, 1, \alpha, \beta\}$, where $\alpha = \alpha'$, $\beta = \beta'$. Define the set

$$S_2 \otimes S_2 = \{0 \otimes 0\} \cup \{a \otimes b\colon a, b \in S_2, a, b \neq 0\} \cup \{(a \otimes b)'\colon a, b \in S_2, a, b \neq 0, 1\}$$

For $\gamma \in S_2 \otimes S_2$ define $0 \otimes 0 \leq \gamma \leq 1 \otimes 1$ and for $a, b \in \{\alpha, \beta\}$ define $a \otimes b \leq a \otimes 1$, $1 \otimes b \leq (a \otimes b)'$. Define $(0 \otimes 0)' = 1 \otimes 1$, $(1 \otimes 1)' = 0 \otimes 0$, and for $a \in \{\alpha, \beta\}$ define $(1 \otimes a)' = 1 \otimes a$, $(a \otimes 1)' = a \otimes 1$. Finally, for $a, b \in \{\alpha, \beta\}$ define the $'$ of $a \otimes b$ as the notation indicates.

Theorem 8.5. The tensor product of S_2 and S_2 is $(S_2 \otimes S_2, \tau_2)$, where $\tau_2(a, 0) = \tau_2(0, a) = 0 \otimes 0$ for all $a \in S_2$ and $\tau_2(a, b) = a \otimes b$ for all a, $b \neq 0$.

Proof. It is straightforward to show that $S_2 \otimes S_2$ is a closed SOP (drawing the Hasse diagram helps) and it is clear that $\tau_2(S_2 \times S_2)$ generates $S_2 \otimes S_2$. Since $a \leq b$ implies that $a \otimes c \leq b \otimes c$ for all $c \in S_2$, we have $\tau_2(a, c) \leq \tau_2(b, c)$ for all $c \in S_2$. Since $a \otimes b \perp a' \otimes c$ for all $b, c \in S_2$, we have $\tau_2(a, b) \perp \tau_2(a', c)$ for all $b, c \in S_2$. Moreover, $\tau_2(a', 1) = \tau_2(a, 1)'$ for all $a \in S_2$. By symmetry, we conclude that τ_2 is a bimorphism. Let α: $S_2 \times S_2 \to R$ be a bimorphism, where R is a closed SOP. Define ϕ: $S_2 \otimes S_2 \to R$ by $\phi(a \otimes b) = \alpha(a, b)$ and $\phi[(a \otimes b)'] = \alpha(a, b)'$. Then $\alpha = \phi \circ \tau_2$ and we must show that ϕ is a morphism. If $a, b \neq 0, 1$, then

$$\phi[(a \otimes b)'] = \alpha(a, b)' = [\phi(a \otimes b)]'$$

and

$$\begin{aligned}\phi[(a \otimes 1)'] &= \phi(a \otimes 1) = \alpha(a, 1) = \alpha(a', 1)\\ &= [\alpha(a, 1)]' = [\phi(a \otimes 1)]'\end{aligned}$$

Suppose $a \otimes b \leq c \otimes d$. Then $a \leq c$ and $b \leq d$, so $\alpha(a, b) \leq \alpha(c, d)$. Hence, $\phi(a \otimes b) \leq \phi(c \otimes d)$. Finally, suppose $a \otimes b \leq (c \otimes d)'$. Then $a \perp c$ or $b \perp d$, so $\alpha(a, b) \perp \alpha(c, d)$. Hence,

$$\phi(a \otimes b) = \alpha(a, b) \leq \alpha(c, d)' = \phi[(c \otimes d)'] \quad \blacksquare$$

Continuing with this program, we have the sharp SOP $S_3 = \{0, 1, \alpha, \alpha', \alpha''\}$. Define the set

$$\begin{aligned}S_3 \otimes S_3 = \{0 \otimes 0\} &\cup \{a \otimes b\colon a, b \in S_3, a, b \neq 0\}\\ &\cup \{(a \otimes b)'\colon a, b \in S_3, a, b \neq 0, 1\}\\ &\cup \{(a \otimes b)''\colon a, b \in S_3, a, b \neq 0, 1\}\end{aligned}$$

Define $(0 \otimes 0)' = 1 \otimes 1$, $(1 \otimes 1)' = 0 \otimes 0$, and for $a \in \{\alpha, \alpha', \alpha''\}$ define $(1 \otimes a)' = 1 \otimes a'$, $(a \otimes 1)' = a' \otimes 1$. Finally for $a, b \in \{\alpha, \alpha', \alpha''\}$ define the $'$ and the $''$ of $a \otimes b$ as notation indicates. We next define $\leq$ on $S_3 \otimes S_3$ as follows:

(1) $a \otimes b \leq c \otimes d$ if $a \leq c$ and $b \leq d$.
(2) $(a \otimes b)' \leq c \otimes d$ if a or $b = 1$ and this reduces to (1).

(3) $a \otimes b \leq (c \otimes d)'$ if $a \leq c'$ or $b \leq d'$.
(4) $(a \otimes b)' \leq (c \otimes d)'$ if $c \otimes b \leq a \otimes b$.
(5) $(a \otimes b)' \leq (c \otimes d)''$ if $(c \otimes d)' \leq a \otimes b$.
(6) $(a \otimes b)'' \leq (c \otimes d)'$ if $c \otimes d \leq (a \otimes b)'$.
(7) $(a \otimes b)'' \leq (c \otimes d)''$ if $a \otimes b \leq c \otimes d$.
(8) $a \otimes b \leq (c \otimes d)''$ if $a \otimes b \leq c \otimes d$.

The proof of the next lemma is similar to (but more involved than) the proof of Lemma 8.5.

Lemma 8.6. The tensor product of S_3 and S_3 is $(S_3 \otimes S_3, \tau_3)$, where $\tau_3(a, 0) = \tau_3(0, a) = 0 \otimes 0$ for all $a \in S_3$ and $\tau_3(a, b) = a \otimes b$ for $a, b \neq 0$.

We also have similar definitions and results for the complementing SOP S_4.

9. TENSOR PRODUCTS OF SOPs

We now give concrete representations of the tensor products of various types of SOPs. This is unlike the orthoalgebra or effect algebra theory, where tensor products are constructed abstractly in terms of the set of all bimorphisms (Dvurečenskij, 1995; Foulis and Bennett, 1993). If $P \subseteq Q^X$ is a Q-function SOP and Q has a certain type, then it is easy to see that P has this same type.

Theorem 9.1. If $P \subseteq Q^X$ and $R \subseteq Q^Y$ are Q-function SOPs, where $Q = S_0, S_1, S_2, S_3$, or S_4, then the tensor product of P and R exists.

Proof. We denote the tensor product of Q and Q by $(Q \otimes Q, \beta)$ and use the notation $\beta(a, b) = a \otimes b$. Now $L = (Q \otimes Q)^{X \times Y}$ is a $Q \otimes Q$-function SOP of the same type as Q (and hence P and R). For $f \in P$, $g \in R$, define $f \otimes g \in L$ by $f \otimes g(x, y) = f(x) \otimes g(y)$ and define $\tau\colon P \times R \to L$ by $\tau(f, g) = f \otimes g$. To show that τ is a bimorphism, suppose $f, g \in P$, $h \in R$ with $f \leq g$. Then $f(x) \leq g(x)$ for all $X \in x$, so $f(x) \otimes h(y) \leq g(x) \otimes h(y)$ for all $x \in X$, $y \in Y$. Hence,

$$\tau(f, h) = f \otimes h \leq g \otimes h = \tau(g, h)$$

If $f \in P$, $g, h \in R$, then $f(x) \otimes g(y) \perp f(x)' \otimes h(y)$ for all $x \in X$, $y \in Y$. It follows that

$$\tau(f, g) = f \otimes g \perp f' \otimes h = \tau(f', h)$$

Moreover, we have

$$\tau(f', 1) = f' \otimes 1 = (f \otimes 1)' = \tau(f, 1)'$$

By symmetry, it follows that τ is a bimorphism. Letting

$$T = \{f \otimes g, (f \otimes g)', (f \otimes g)'': f \in P, g \in R\}$$

we conclude that T is the sub-SOP of L generated by $\tau(P \times R)$.

To show that (T, τ) is the tensor product of P and R, let $\alpha: P \times R \to V$ be a bimorphism, where V is a SOP of the same type as Q. Define $\phi: T \to V$ by $\phi(f \otimes g) = \alpha(f, g)$, $\phi((f \otimes g)') = \alpha(f, g)'$, and $\phi((f \otimes g)'') = \alpha(f, g)''$. Then ϕ preserves $'$ and $\alpha = \phi \circ \tau$. To show that ϕ is a morphism, suppose $f \otimes g \leq h \otimes i$. Then $f \leq h$ and $g \leq i$, so

$$\phi(f \otimes g) = \alpha(f, g) \leq \alpha(h, i) = \phi(h \otimes i)$$

Next, suppose that $f \otimes g \leq (h \otimes i)'$. Then

$$f(x) \otimes g(y) \leq [h(x) \otimes i(y)]'$$

for every $x \in X$, $y \in Y$. If $g(y_0) \not\perp i(y_0)$ for some $y_0 \in Y$, then

$$f(x) \otimes g(y_0) \leq [h(x) \otimes i(y_0)]'$$

for every $x \in X$. It follows that $f \perp h$. Hence, either $f \perp h$ or $g \perp i$, so

$$\phi(f \otimes g) = \alpha(f, g) \leq \alpha(h, i)' = \phi((h \otimes i)')$$

The other cases follow in a similar way. For example, suppose that $(f \otimes g)' \leq (h \otimes i)'$. It follows that

$$h(x) \otimes i(y) \leq f(x) \otimes g(y)$$

for all $x \in X$, $y \in Y$, so $h \leq f$ and $i \leq g$. Hence, $\alpha(h, i) \leq \alpha(f, g)$, so $\alpha(f, g)' \leq \alpha(h, i)'$. Hence,

$$\phi((f \otimes g)') = \alpha(f, g)' \leq \alpha(h, i)' = \alpha((h \otimes i)') \quad \blacksquare$$

Corollary 9.2. Let $P \subseteq Q^X$ and $R \subseteq Q^Y$ be Q-function SOPs, where $Q = S_0, S_1, S_2, S_3$, or S_4. Then the tensor product $(P \otimes R, \tau)$ of P and R is a sub-SOP of $(Q \otimes Q)^{X \times Y}$ and $\tau(f, g)(x, y) = f(x) \otimes g(y)$ for all $f \in P$, $g \in R$, $x \in X$, $y \in Y$.

To be specific, we say that a SOP P has *type* 0, 1, 2, 3, or 4 if P is: an orthoposet; closed and regular; closed, sharp, and unit; or complementing, respectively.

Corollary 9.3. Let P and R be SOPs of type t, $t = 0, 1, 2, 3, 4$. Then the tensor product $(P \otimes R, \tau)$ of P and R is an $S_t \otimes S_t$-function SOP. Moreover, if μ and ν are S_t-morphisms on P and R, respectively, then there exists a unique $S_t \otimes S_t$-morphism λ on $P \otimes R$ such that $\lambda[\tau(a, b)] = \mu(a) \otimes \nu(b)$ for all $a \in P$, $b \in R$.

Proof. Let M_P and M_R be the set of all S_t-morphisms on P and R, respectively. From previous results, we have seen that there exist isomorphisms u, v from P, R to S_t-function SOPs $F_P \subseteq S_t^{M_P}$, $F_R \subseteq S_t^{M_R}$, respectively. Let $(F_P \otimes F_T, \tau_1)$ be the tensor product of F_P and F_R in accordance with Corollary 9.2. Define τ: $P \times R \to F_P \otimes F_R$ by

$$\tau(a, b) = \tau_1(u(a), v(b)) = u(a) \otimes v(b)$$

We easily conclude that τ is a bimorphism and $\tau(P \times R)$ generates $F_P \otimes F_R$. Let α: $P \times R \to L$ be a bimorphism, where L has type t. It follows that α_1: $F_P \times F_R \to L$ given by $\alpha_1(u(a), v(b)) = \alpha(a, b)$ is a bimorphism. Hence, there exists a morphism ϕ: $F_P \otimes F_R \to L$ such that $\alpha_1 = \phi \circ \tau_1$. Now

$$(\phi \circ \tau)(a, b) = \phi[\tau_1(u(a), v(b))] = \alpha_1(u(a), v(b)) = \alpha(a, b)$$

Hence, $\alpha = \phi \circ \tau$, so $(F_P \otimes F_R, \tau)$ is the tensor product of P and R. Finally, let μ and ν be S_t-morphisms on P and R, respectively. Define λ: $F_P \otimes F_R \to S_t \otimes S_t$ by $\lambda(f \otimes g) = f(\mu) \otimes g(\nu)$. It is clear that λ is an $S_t \otimes S_t$-morphism and we have

$$\lambda[\tau(a, b)] = \lambda[u(a) \otimes v(b)] = u(a)(\mu) \otimes v(b)(\nu)$$

$$\mu(a) \otimes \nu(b)$$

Finally, λ is clearly unique. ∎

Corollary 9.4. Let P and R be orthoposets. Then the tensor product $(P \otimes R, \tau)$ of P and R is a standard set-SOP of subsets of a Cartesian product $X \times Y$ and $\tau(a, b) = A \times B$, where $A \subseteq X$, $B \subseteq Y$. Moreover, if μ and ν are S_0-morphisms on P and Q, respectively, then there exists a unique $\lambda \in X \times Y$ such that $\mu(a)\nu(b) = \chi_{A \times B}(\lambda)$ for all $a \in P$, $b \in R$.

Corollary 9.5. Let P and R be closed, regular SOPs. Then the tensor product $(P \otimes R, \tau)$ of P and R is a $[0, 1]$-function SOP of functions on a Cartesian product $X \times Y$ and $\tau(a, b)(x, y) = f(x)g(y)$, where $f \in [0, 1]^X$, $g \in [0, 1]^Y$. Moreover, if μ and ν are $[0, 1]$-morphisms on P and Q, respectively, then there exists a unique $(x, y) \in X \times Y$ such that $\mu(a)\nu(b) = \tau(a, b)(x, y)$ for all $a \in P$, $b \in R$.

REFERENCES

Beltrametti, E., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics,* Addison-Wesley, Reading, Massachusetts.

Birkhoff, G., and von Neumann, J. (1936). The logic of quantum mechanics, *Annals of Mathematics,* **37**, 823–843.

Cattaneo, G., and Nistico, G. (1989). Brouwer–Zadeh posets and three-valued Lukasiewicz posets, *International Journal of Fuzzy Sets and Systems*, **33**, 165–190.

Dalla Chiara, M. L., and Giuntini, R. (1989). Paraconsistent quantum logics, Foundations of Physics, **19**, 891–904.

Dvurečenskij, A. (1995). Tensor product of difference posets, *Transactions of the American Mathematical Society*, **347**, 1043–1057.

Dvurečenskij, A. (n.d.). Measures and ⊥-decomposable measures on effects of a Hilbert space, to appear.

Dvurečenskij, A., and Riečan, B. (1994). Decompositions of measures on orthoalgebras and difference posets, *International Journal of Theoretical Physics*, **33**, 1403–1418.

Foulis, D., and Bennett, M. K. (1993). Tensor products of orthoalgebras, *Order*, **10**, 271–282.

Foulis, D., and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics, *Foundations of Physics*, **24**, 1331–1352.

Foulis, D., and Bennett, M. K. (n.d.). Sums and products of interval algebras, to appear.

Foulis, D., and Randall, C. (1981). Empirical logic and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., Wissenschaftsverlag, Bibliographisches Institute 5, Mannheim, pp. 9–20.

Foulis, D., Greechie, R., and Rüttimann, G. (1992). Filters and supports in orthoalgebras, *International Journal of Theoretical Physics*, **31**, 789–807.

Giuntini, R. (1990). Brouwer–Zadeh logic and the operational approach to quantum mechanics, *Foundations of Physics*, **20**, 701–715.

Giuntini, R., and Greuling, H. (1989). Toward a formal language for unsharp properties, *Foundations of Physics*, **19**, 931–945.

Gudder, S. (1979). *Stochastic Methods in Quantum Mechanics*, North-Holland, New York.

Gudder, S. (1988). *Quantum Probability*, Academic Press, New York.

Jauch, J. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Katrnoška, F. (1982). On the representation of orthocomplemented posets, *Commentationes Mathematicae Universitatis Carolinae*, **23**, 489–498.

Kläy, M., Randall, C. and Foulis, D. (1987). Tensor products and probability weights, *International Journal of Theoretical Physics*, **26**, 199–216.

Kôpka, F., and Chovanec, F. (1994). D-posets, *Mathematica Slovaca*, **44**, 21–34.

Ludwig, G. (1986). *An Axiomatic Basis for Quantum Mechanics*, Vol. 2, Springer, New York.

Mackey, G. (1963). *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.

Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, New York.

Prugovečki, E. (1986). *Stochastic Quantum Mechanics and Quantum Space Time*, Reidel, Dordrecht.

Ptak, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.

Randall, C., and Foulis, D. (1981). Operational statistics and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., Wissenschaftsverlag, Bibliographisches Institute 5, Mannheim, pp. 21–28.

Schroeck, F., and Foulis, D. (1990). Stochastic quantum mechanics viewed from the language of manuals, *Foundations of Physics*, **20**, 823–858.

Varadarajan, V. (1968/1970). *Geometry of Quantum Theory*, Vols. 1 and 2, Van Nostrand Reinhold, Princeton, New Jersey.